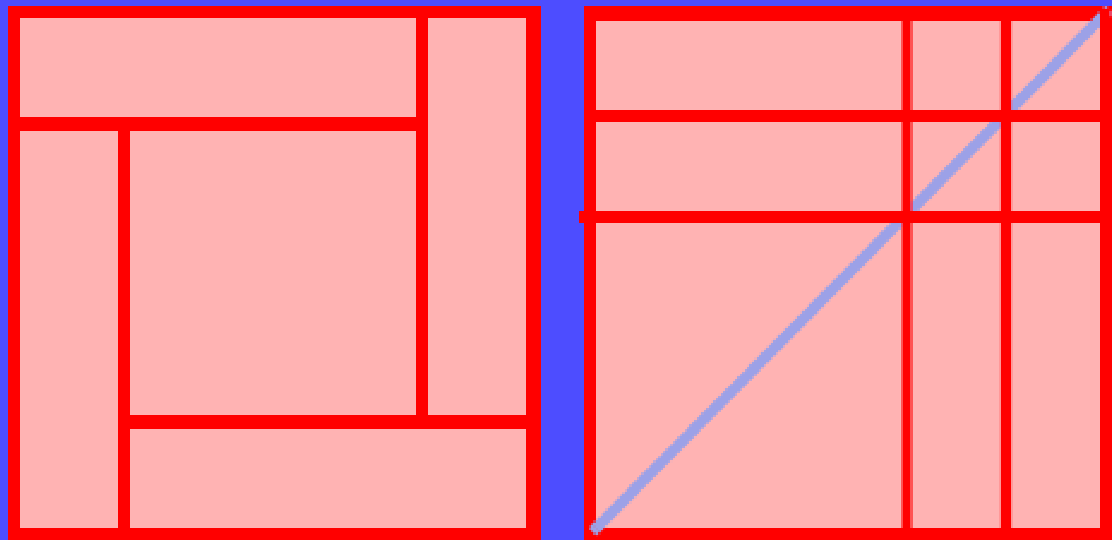
- gured Algebra

Algebra v slikah



KUPM 2018

Martin Kindt
Freudenthal Institute
Utrecht

Matematika spodbuja mišljenje.........
pod pogojem, da se poučuje
in uči ustrezno

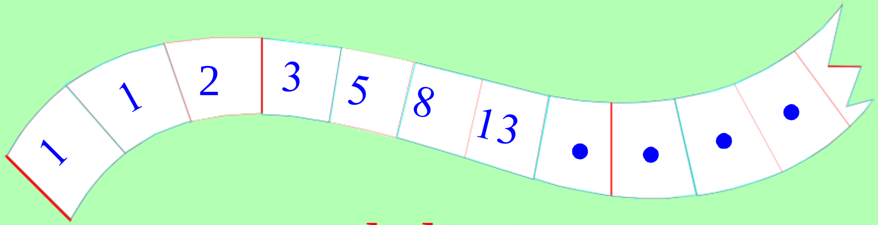
George Polya
1887-1985

ICME 1980 Berkely

1980

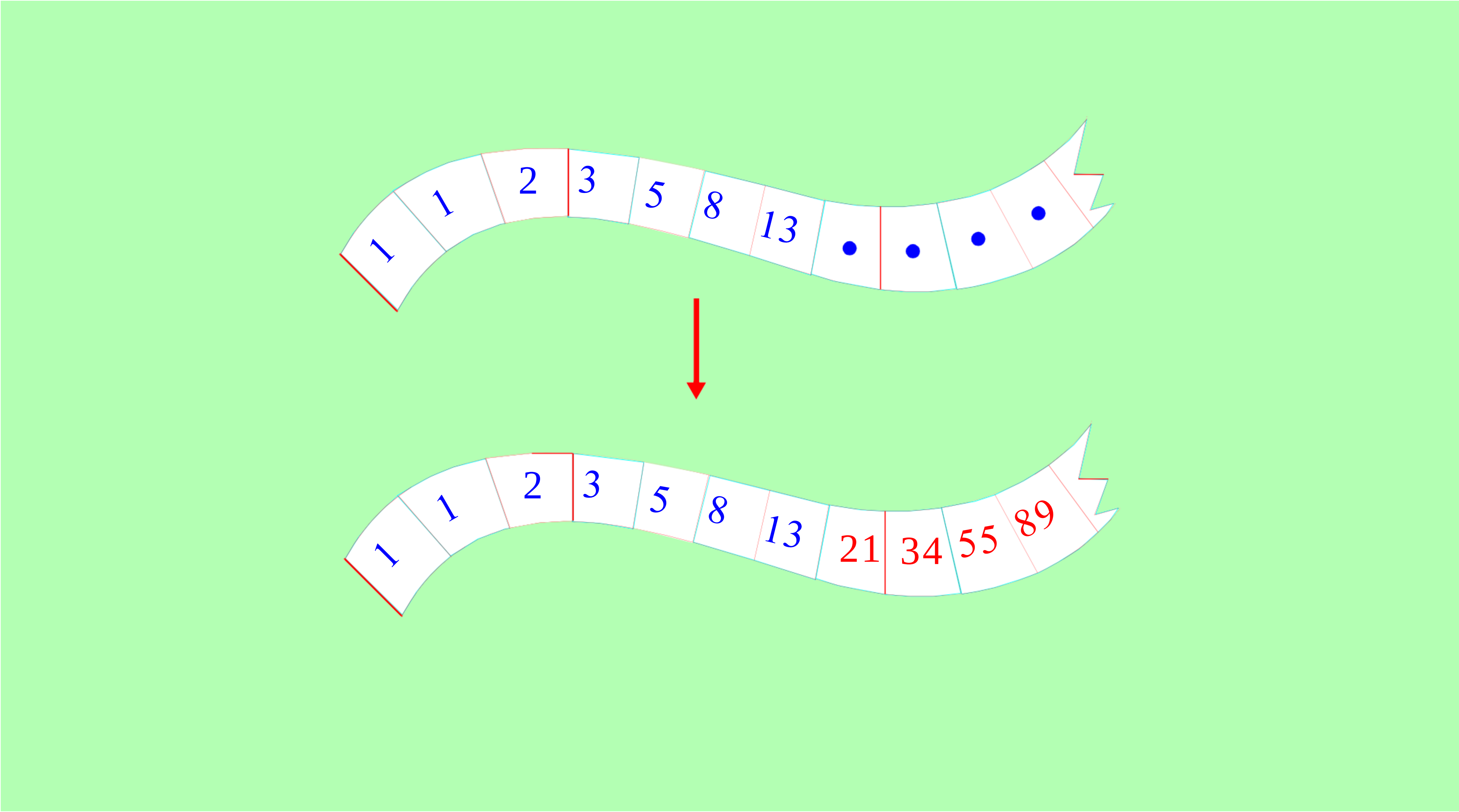
majica iz San Francisca

......

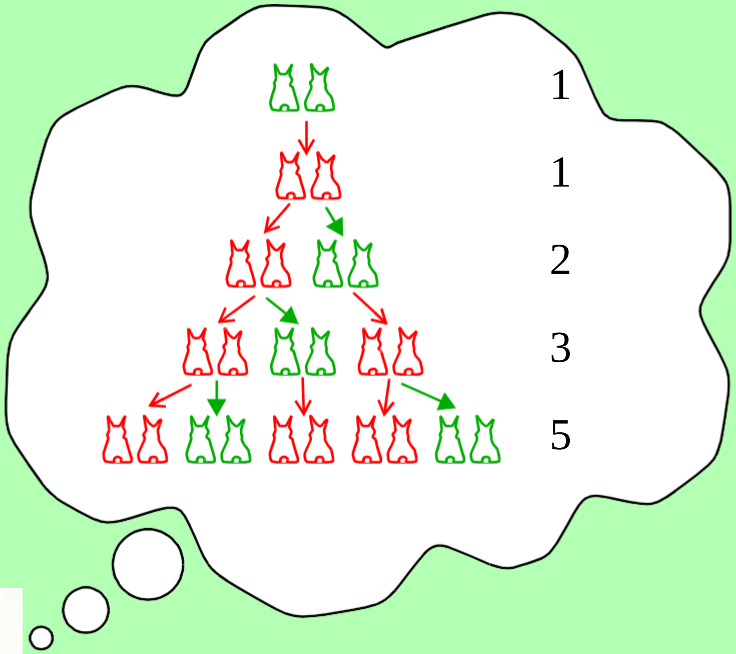


Nadaljuj

....


1
1
2
3
5
8
13
1
1
2
3
5
8
13
21
34
55
89

izviren kontekst



Fibonacci

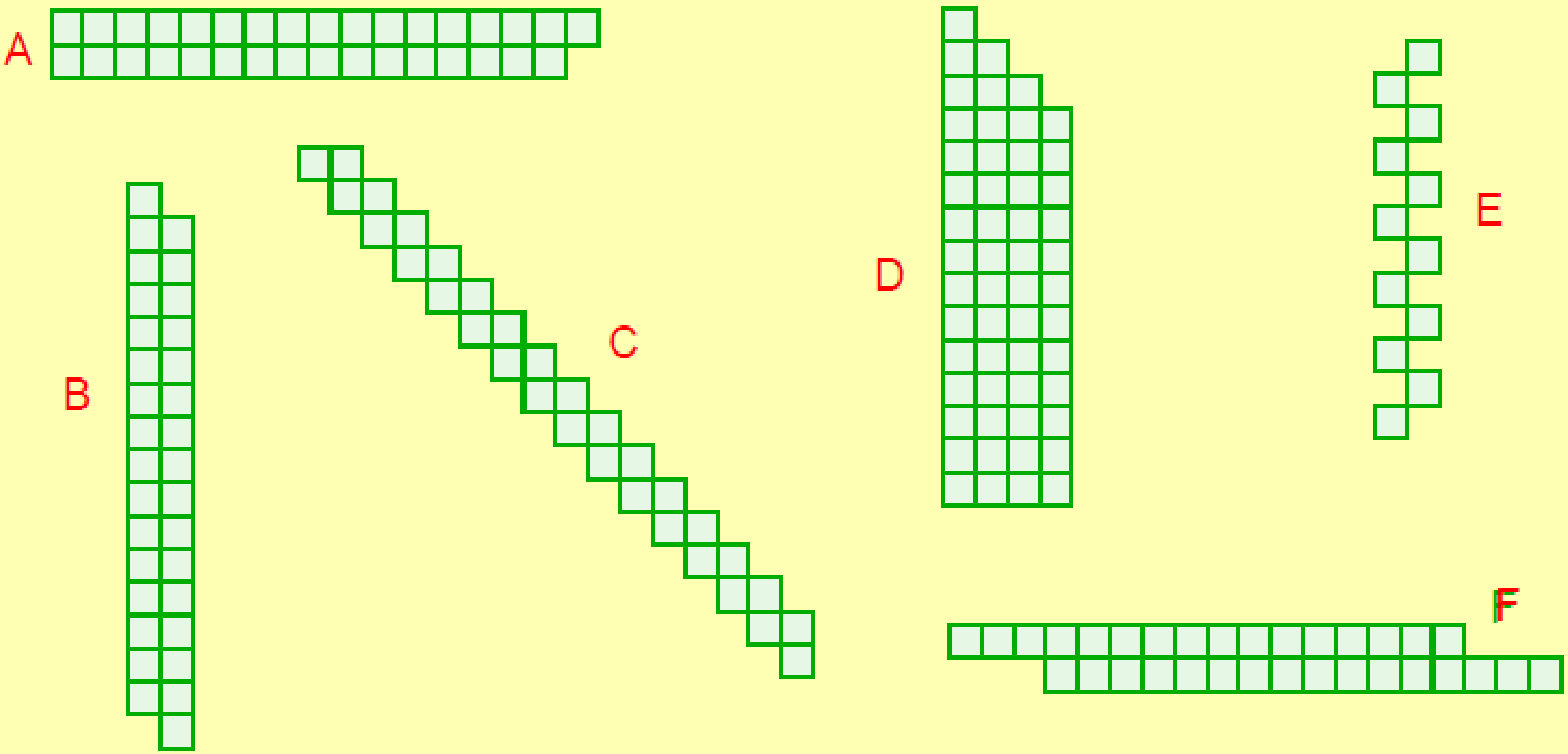
Liber Abaci (1202)

$f_0 = 1$	$f_{10} = 89$	$f_{20} = 10946$
$f_1 = 1$	$f_{11} = 144$	$f_{21} = 17711$
$f_2 = 2$	$f_{12} = 233$	$f_{22} = 28657$
$f_3 = 3$	$f_{13} = 377$	$f_{23} = 46368$
$f_4 = 5$	$f_{14} = 610$	$f_{24} = 75025$
$f_5 = 8$	$f_{15} = 987$	$f_{25} = 121393$
$f_6 = 13$	$f_{16} = 1597$	$f_{26} = 196418$
$f_7 = 21$	$f_{17} = 2584$	$f_{27} = 317811$
$f_8 = 34$	$f_{18} = 4181$	$f_{28} = 514229$
$f_9 = 55$	$f_{19} = 6765$	$f_{29} = 832040$

vzorec znotraj Fibonaccijevega zaporedja?

1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 ...

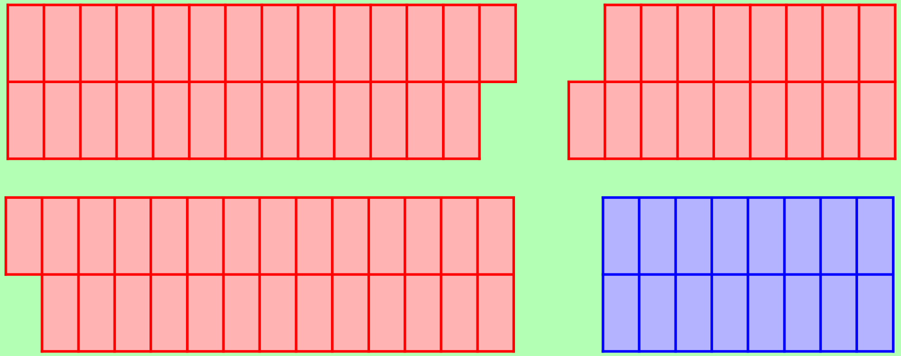
Sodo ali liho? Utemelji izbrano strategijo!



Od 'VZORCEV in SIMBOLOV'

(Mathematics in Context, 1997)

1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 ...



LIHO + LIHO = SODO

LIHO + SODO = LIHO

W.W. Sawyer (1911-2008): ‘Vision in Elementary Mathematics’

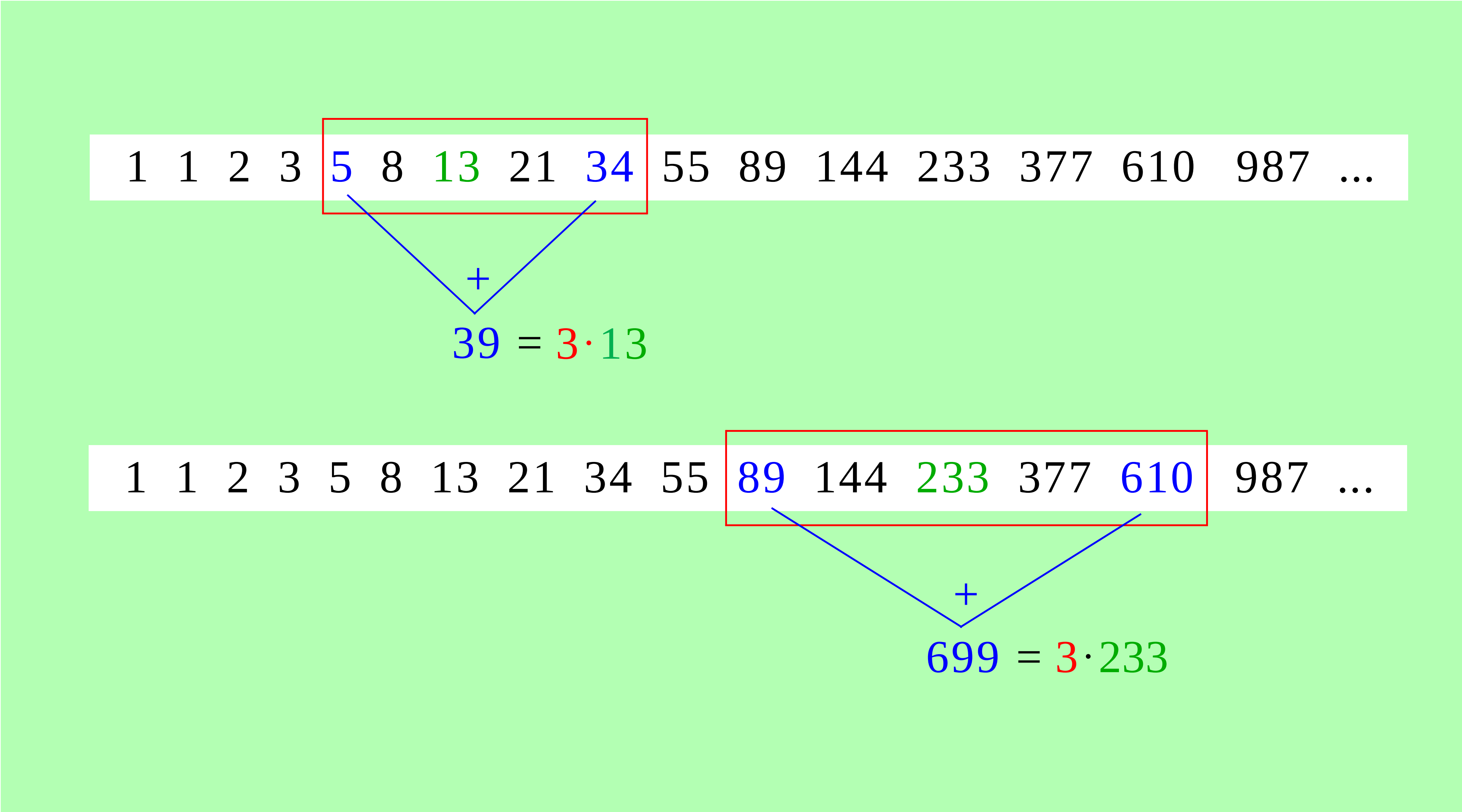
Izberite pet zaporednih Fibonaccijevih števil....

1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 ...

+

$24 = 3 \cdot 8$

Ali velja lastnost za vsakih pet zaporednih Fibonaccijevih števil?


1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 ...
+
39 = 3·13
1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 ...
+
699 = 3·233

reprezentacija poljubnih 5 zaporednih Fibonaccijevih števil

prvi

drugi

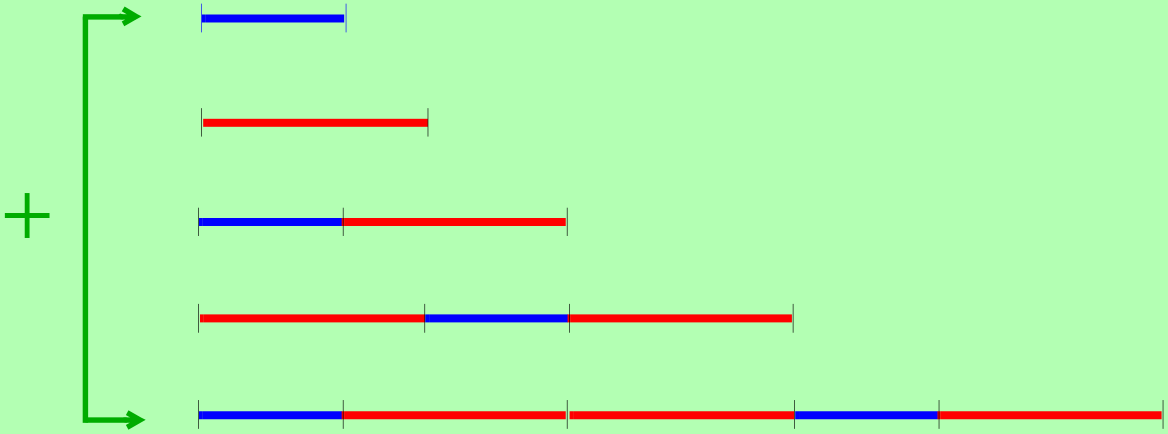
tretji

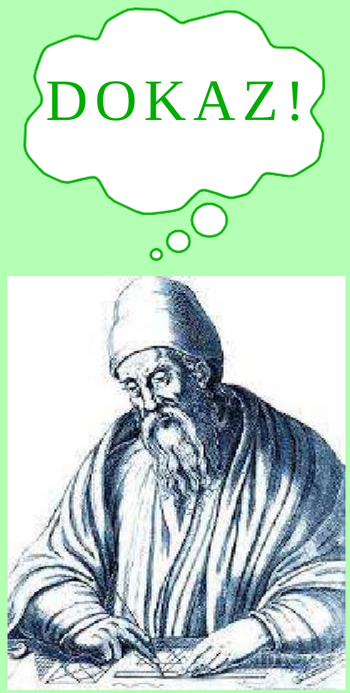
četrti

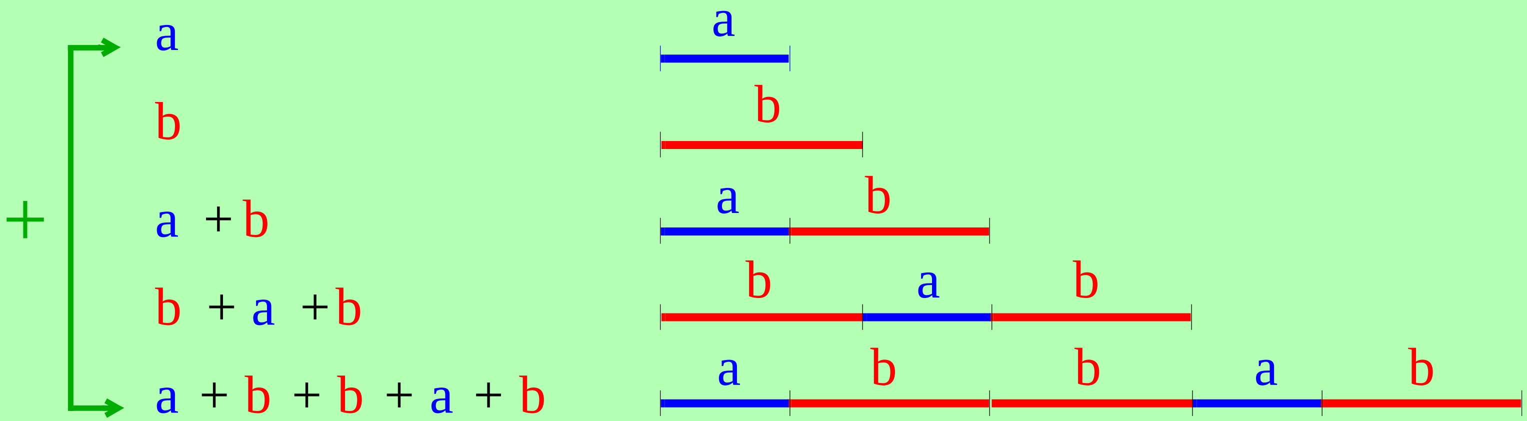
peti

v slogu

Evklida




DOKAZ!



$$a + (a + b + b + a + b) = (a + b) + (a + b) + (a + b)$$

$+$

a

b

$a + b$

$a + 2b$

$2a + 3b$

$$a + (2a + 3b)$$
$$=$$
$$3a + 3b$$
$$=$$
$$3 \cdot (a + b)$$

Do formalnega zapisa v več korakih

‘Fibonaccijeve naloge’

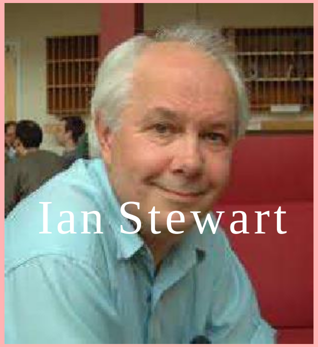
* Izberite poljubno podzaporedje devetih zaporednih števil.
  Vsota prvega in devetega števila je enaka....

1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 ...

* Primerjaj vsoto poljubnih šest zaporednih Fibonaccijevih števil s petim.
  Kaj opaziš? Dokaži, da lastnost velja za poljubnih šest zaporednih Fibonaccijevih števil.

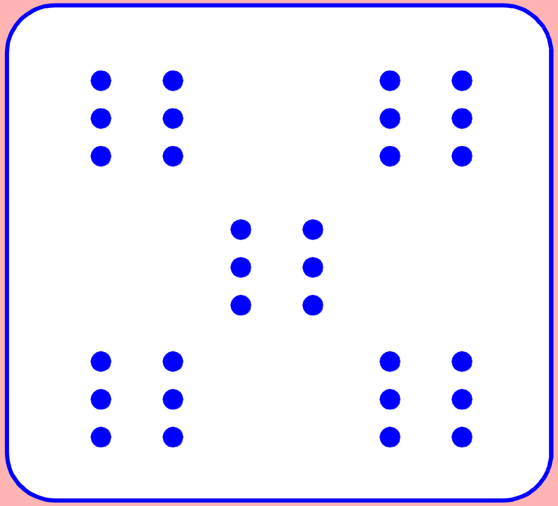
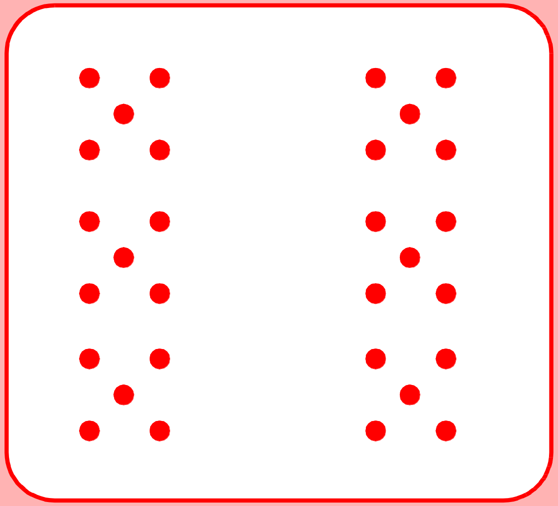
* Sestavi Fibonaccijevo nalogo.

VZORCI
&
SLIKE



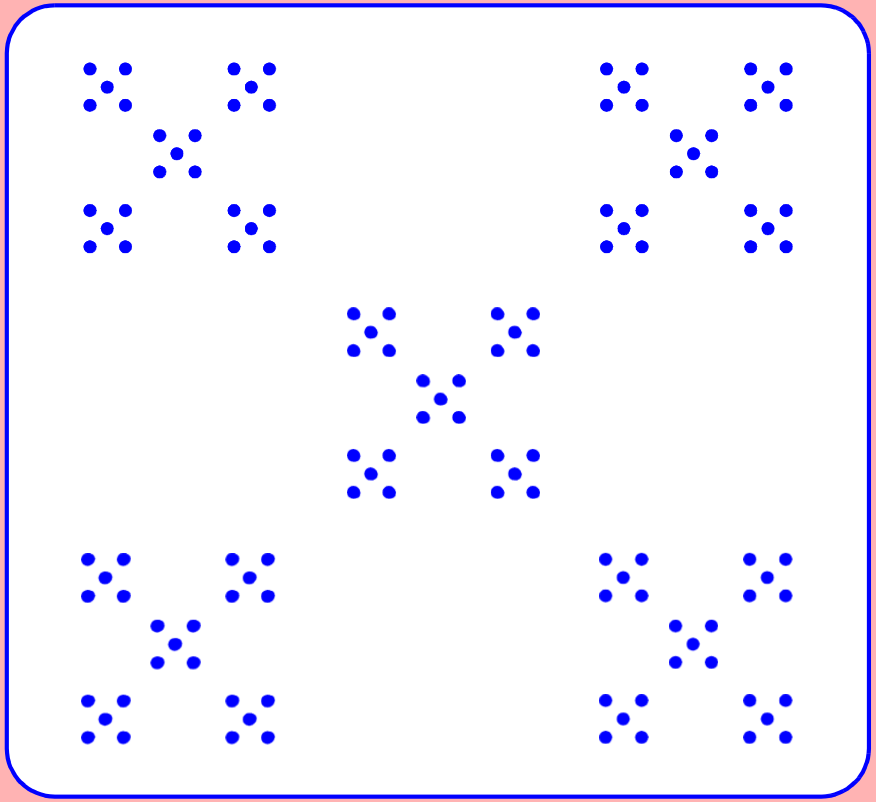
Ideja naravnih števil je že tako dobro in dolgo premišljena in utemeljena, da o njih razmišljamo kot o stvareh.

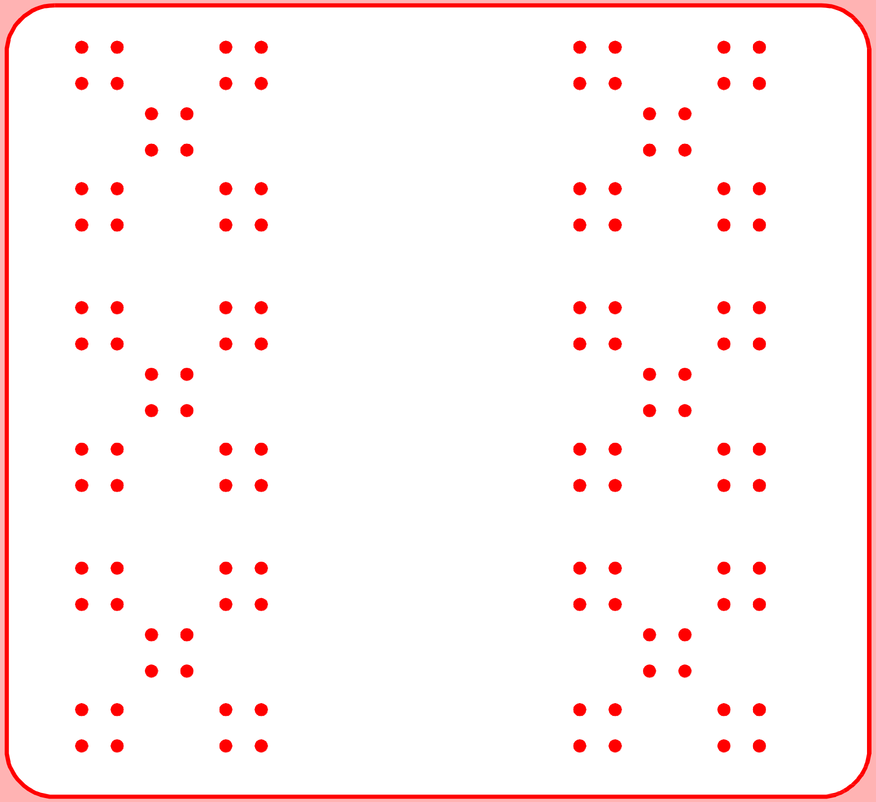
slikovna reprezentacija števil

Kateri vzorec ima večje število pik?

Enako vprašanje....



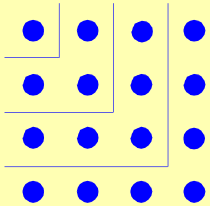


»Produktivne« naloge

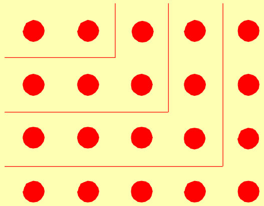
* Iz kock oblikuj »pravilen« vzorec s 625 pikami.
* Iz kock oblikuj vzorec s številom pik med 100 in 1000.
* ....

Nikomachos of Gerasa
(ok. 100 n. št.)

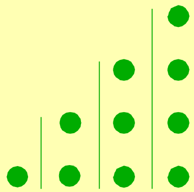
Uvod v
aritmetiko



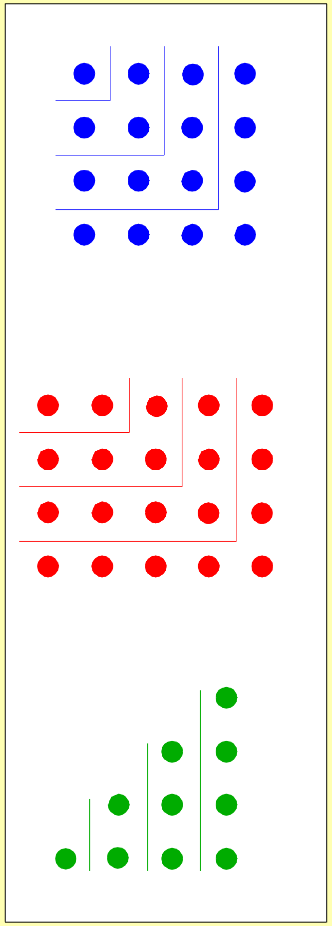
vsota prvih zaporednih lihih števil
=
kvadratno število



vsota prvih zaporednih sodih števil
=
pravokotniško število



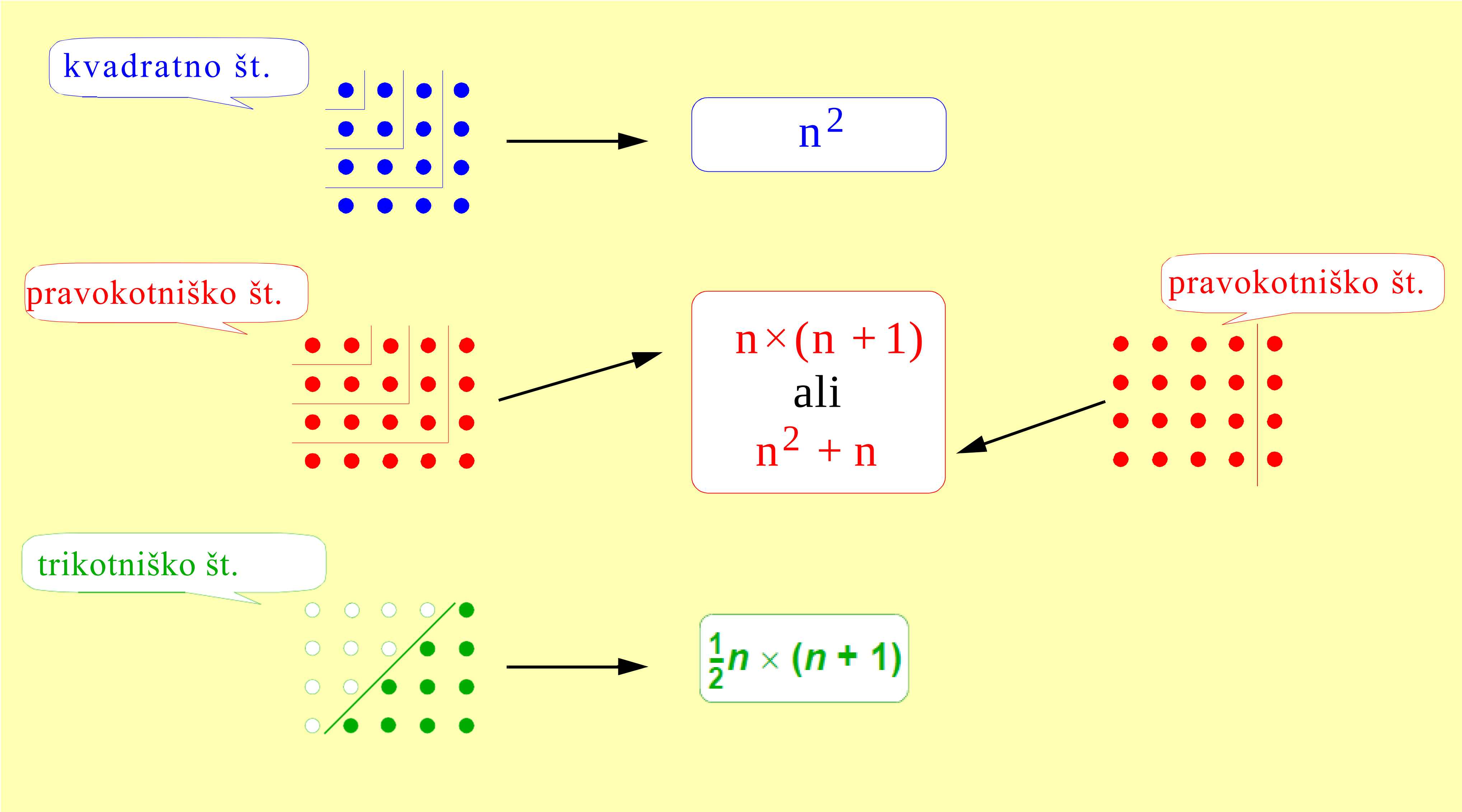
Vsota zaporednih naravnih števil
=
trikotniško število



$$1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$$

$$2 + 4 + 6 + 8 + 10 + 12 = 42 = 6 \cdot 7$$

$$1 + 2 + 3 + 4 + 5 + 6 = 21 = \frac{1}{2} \cdot 6 \cdot 7$$

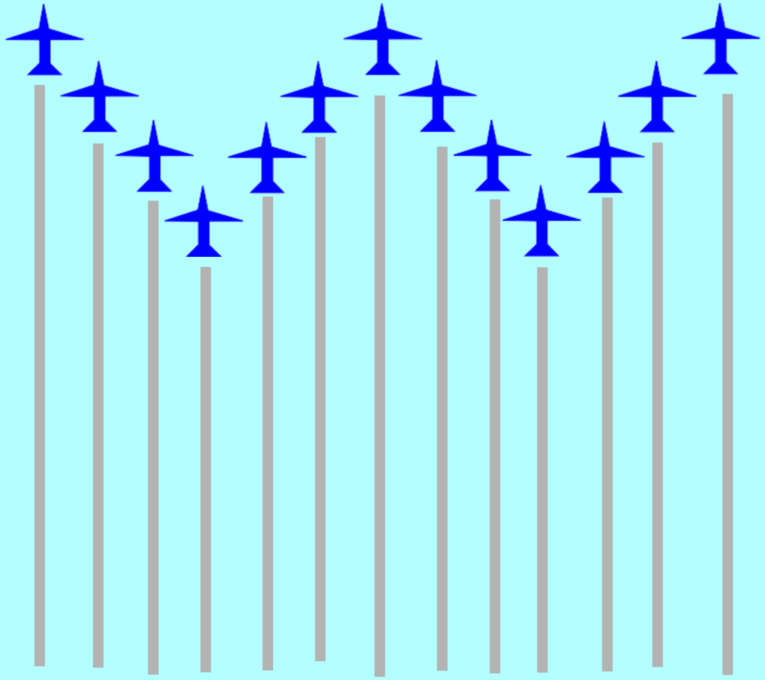


kvadratno št.
n2
pravokotniško št.
n×(n + 1)
ali
n2 + n
pravokotniško št.
trikotniško št.
½n × (n + 1)

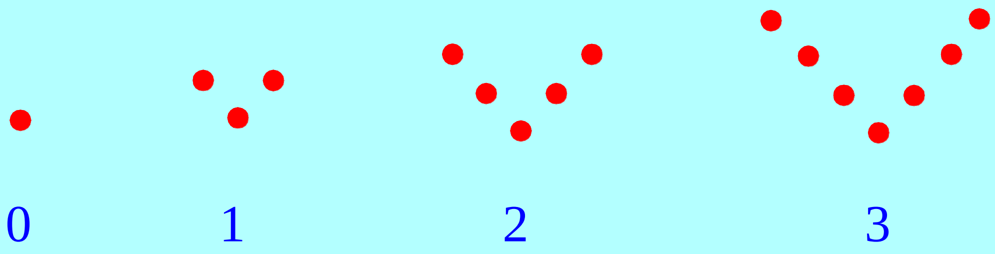
Prelet letal

‘V-postavitev’

‘W-postavitev’



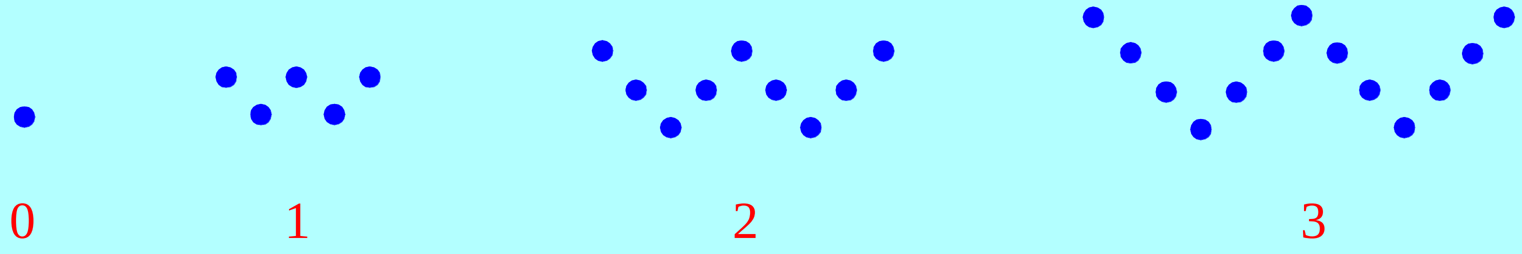
‘V-števila’



zaporedno število	0	1	2	3	4	5	6
število pik	1	3	...	...	...	...	...

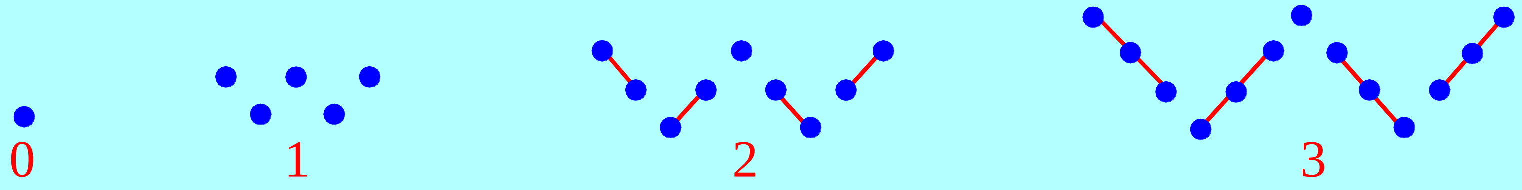
Algebrski izraz?

‘W-števila’

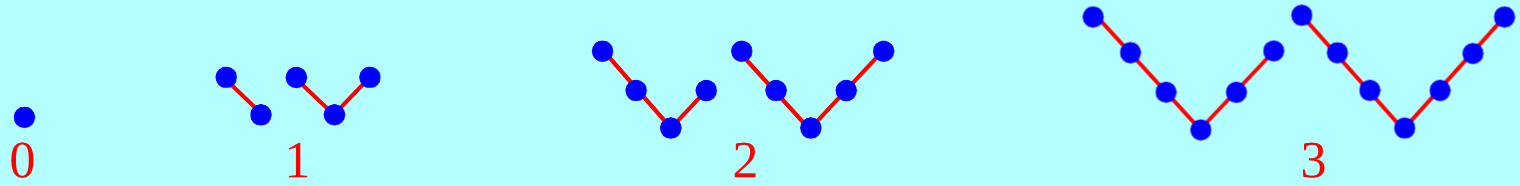


zaporedno število	0	1	2	3	4	5	6
število pik	1	5	...	...	...	...	...

Algebrski izraz?

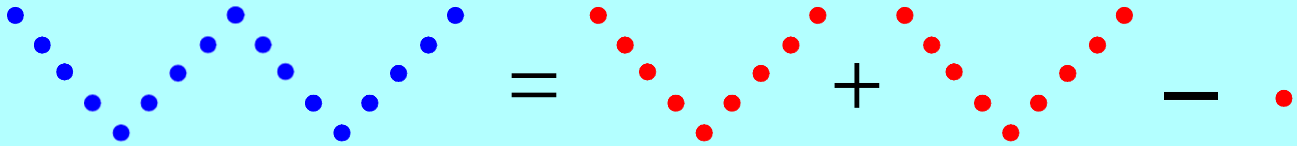


$$W = 4 \cdot n + 1$$

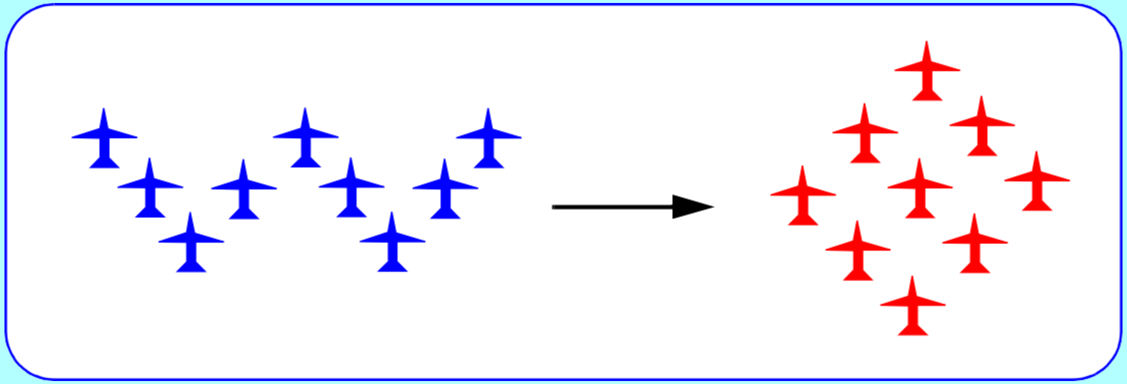


$$W = [2n] + [2n + 1] = 4n + 1$$

$$W = \text{dvojni } V - 1$$



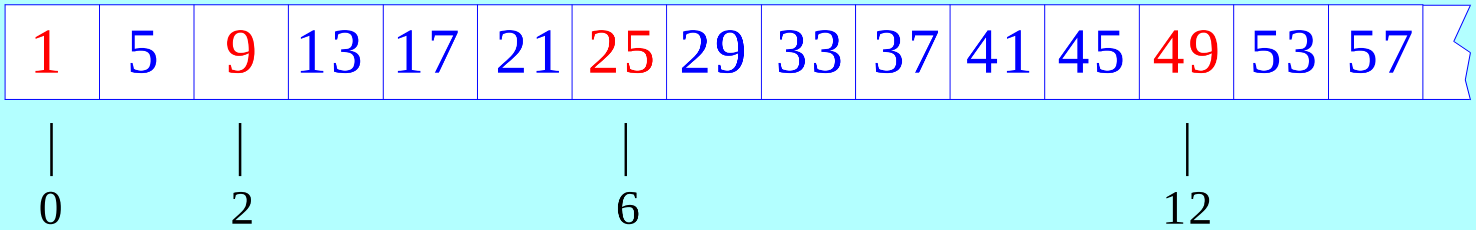
$$W = (1 + 2n) + (1 + 2n) - 1 = 1 + 4n$$



W-postavitve včasih lahko preoblikujejo v „kvadratne“

* Katere? Zakaj?

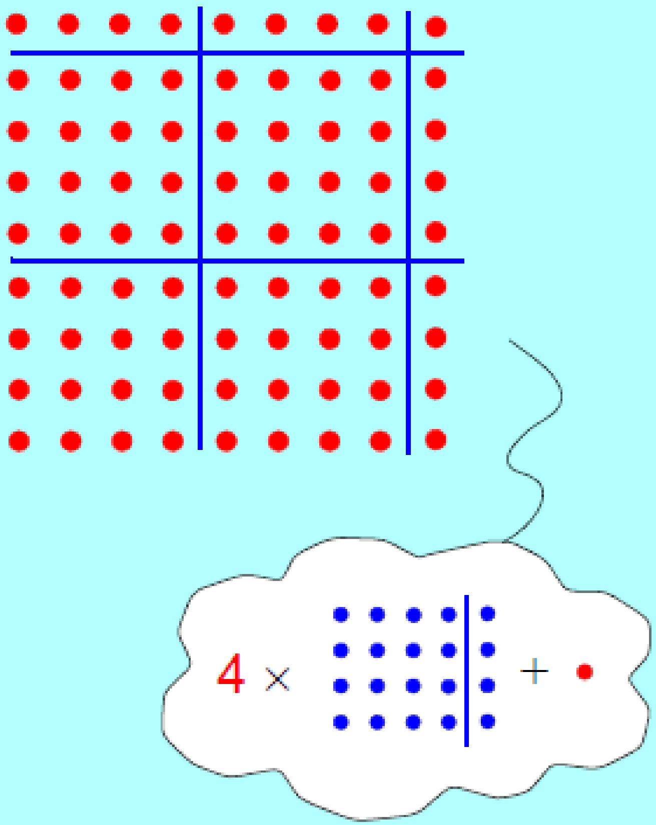
ZaporedjeW-števil



ZaporedjeW-števil

1	5	9	13	17	21	25	29	33	37	41	45	49	53	57	

Vsako liho
kvadratno
število je
W-število ?!



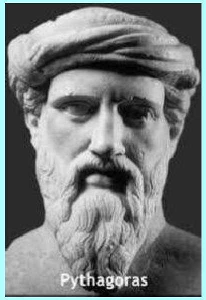
$$\begin{array}{rl} 2n+1 & \\ 2n+1 & \cdot \\ \hline 2n+1 & \\ 4n^2+2n & \\ \hline 4n^2+4n+1 & + \end{array}$$

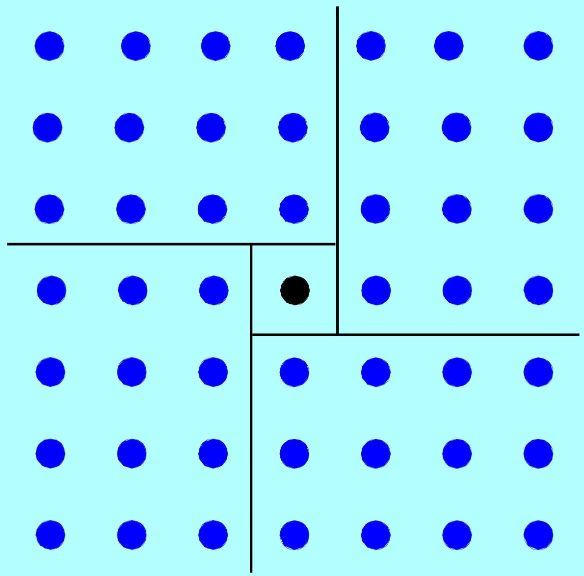
W-število

$$4n^2+4n+1 = \boxed{4(n^2+n)+1}$$

pravokotniško število

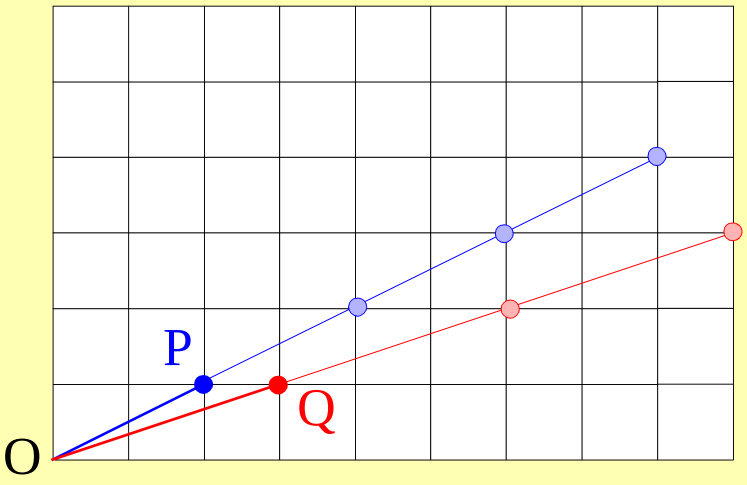
$4\cdot$ pravokotniško število + 1 = kvadratno število



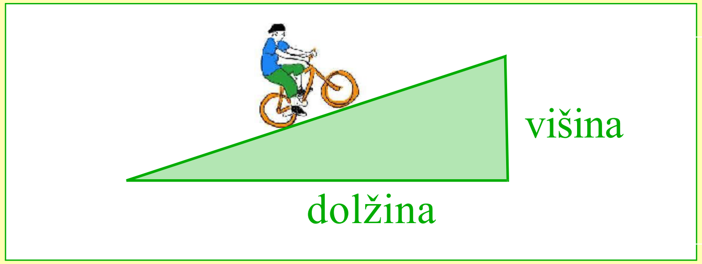


$$4\cdot n(n+1)+1=(n+1)^2$$

Ulomki & Naklon



naklon = $\dfrac{\text{‘višina’}}{\text{‘dolžina’}}$



$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \dots$

$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \dots$

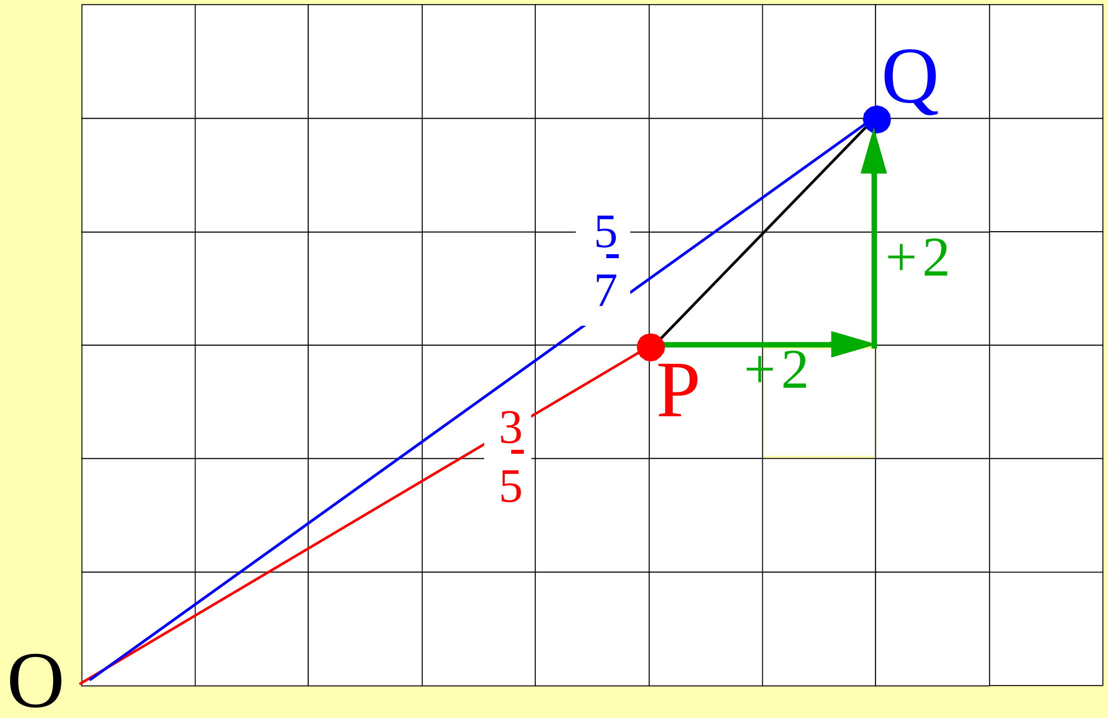
ekvivalentni ulomki

naklon se ne spremeni

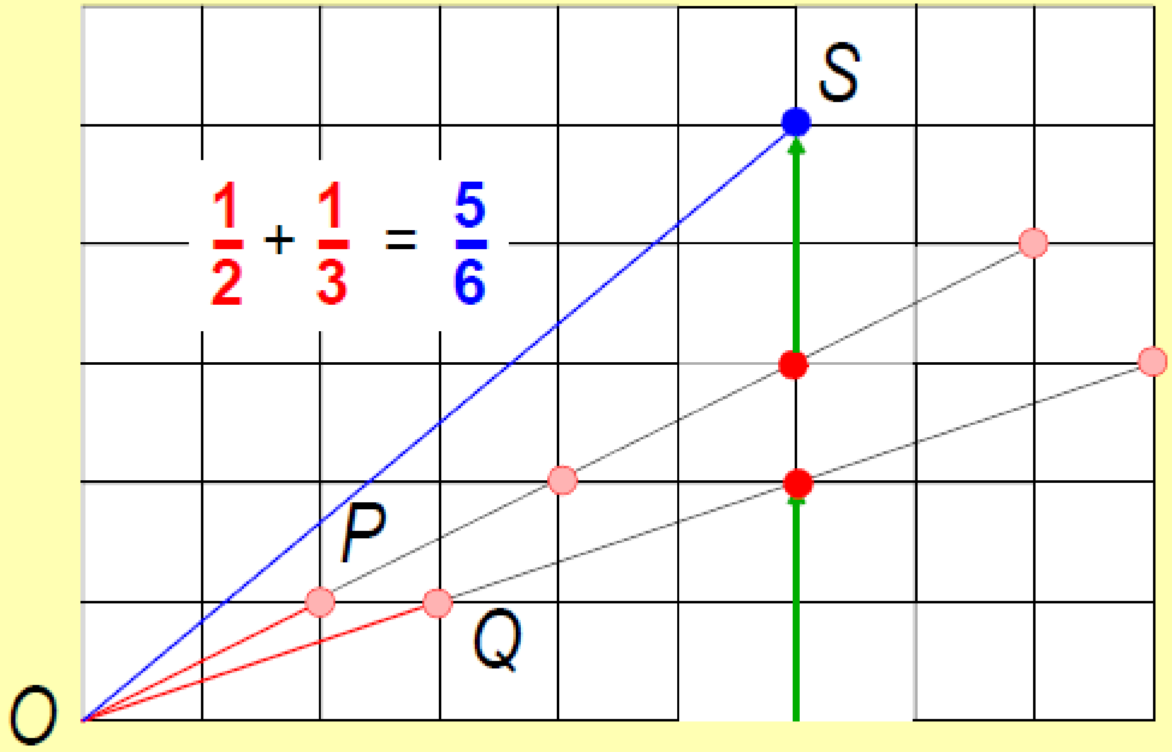
$$\frac{3}{5} = \frac{3 \times 2}{5 \times 2}$$

naklon se poveča

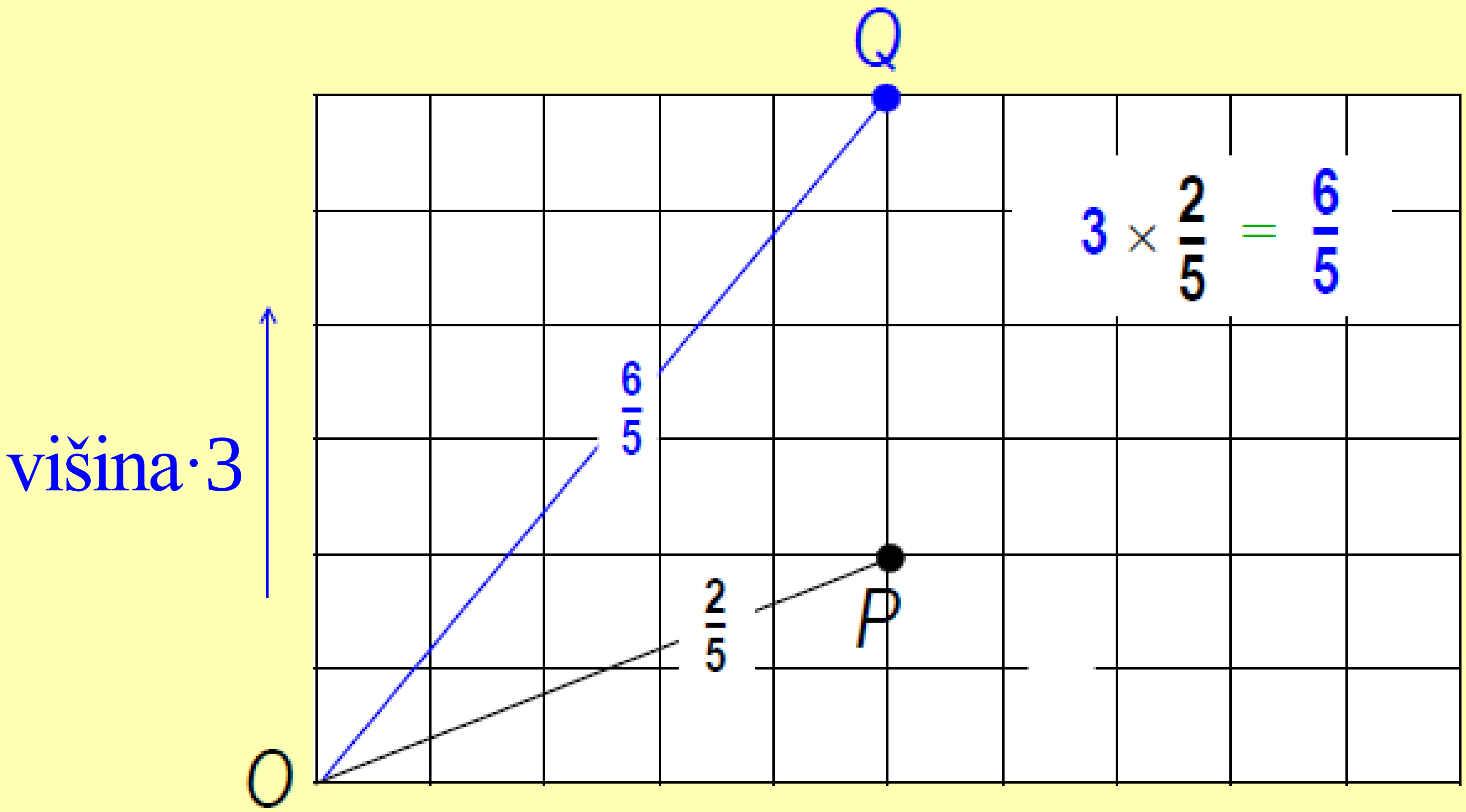
$$\frac{3}{5} < \frac{3 + 2}{5 + 2}$$

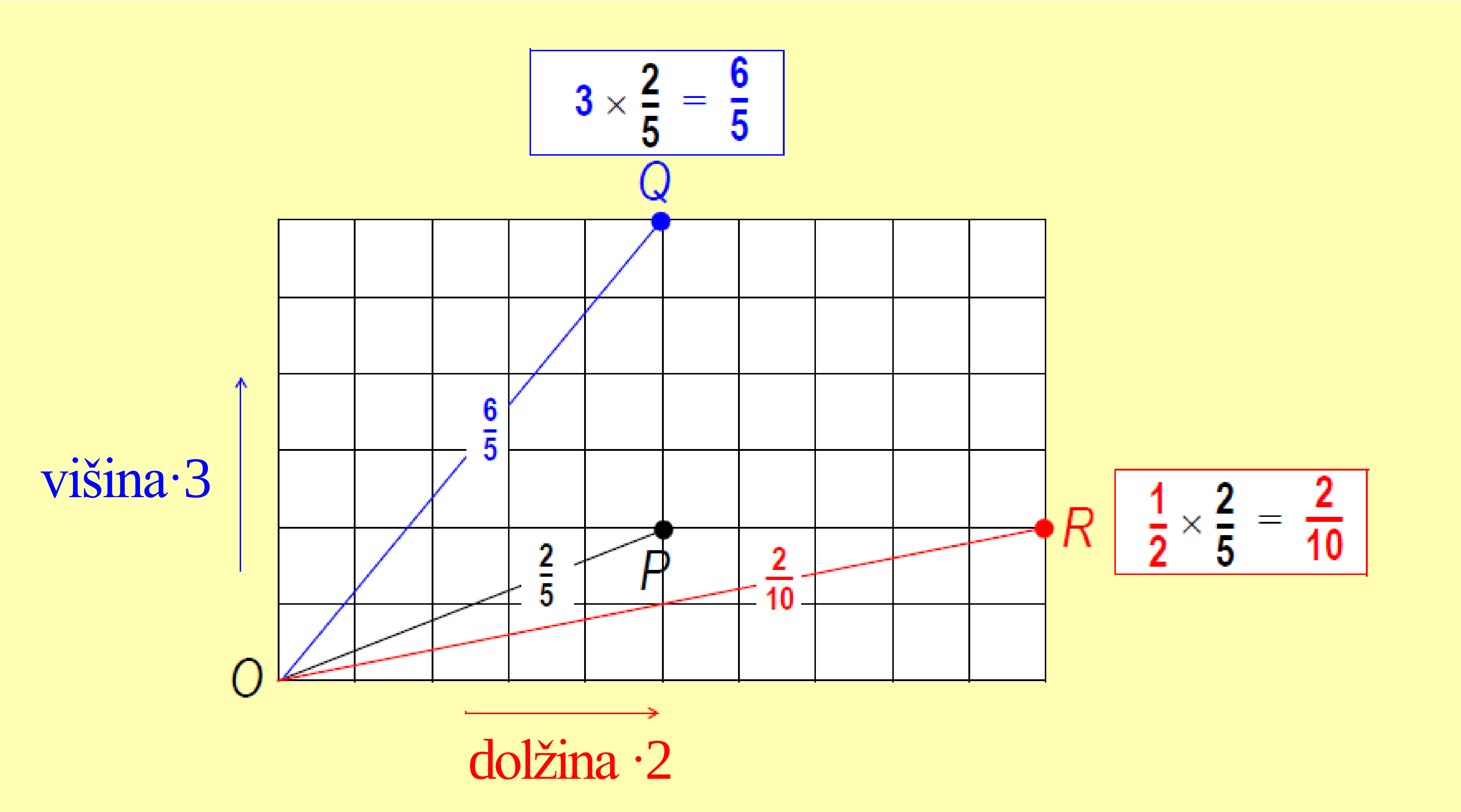


Seštevanje in naklon



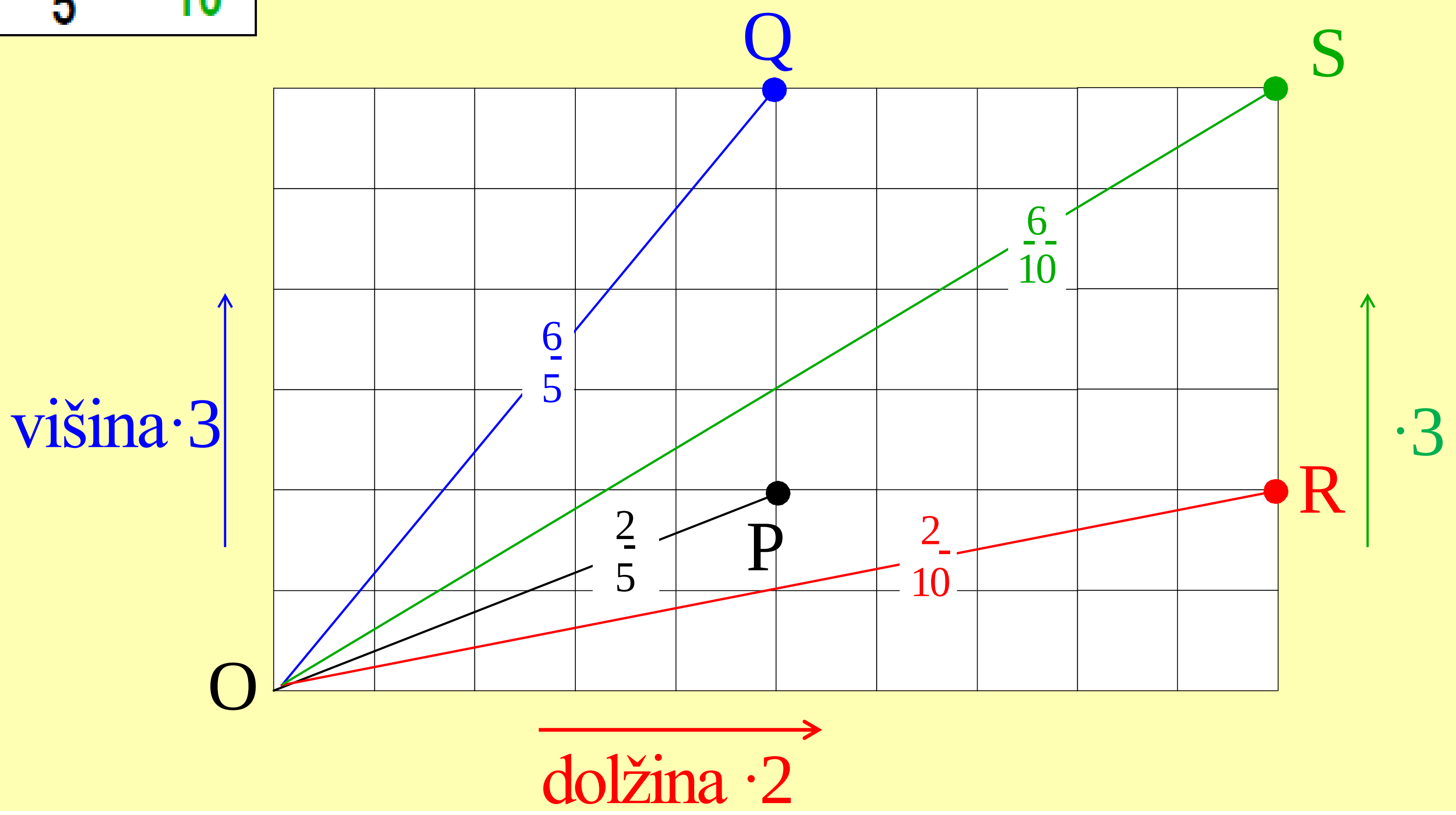
Množenje





$3 \times \frac{2}{5} = \frac{6}{5}$
Q
$\frac{6}{5}$
višina·3
$\frac{2}{5}$
P
$\frac{2}{10}$
R
$\frac{1}{2} \times \frac{2}{5} = \frac{2}{10}$
O
dolžina ·2

$$\frac{3}{2} \times \frac{2}{5} = \frac{6}{10}$$



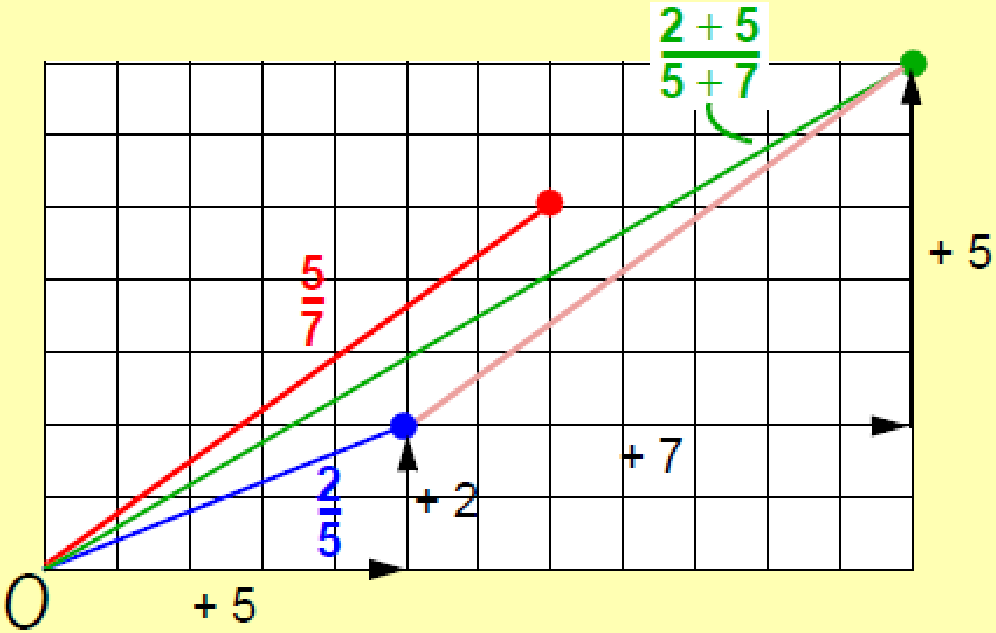
Množenje

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

Seštevanje

$$\frac{a}{b} + \frac{c}{d} \stackrel{?}{=} \frac{a + c}{b + d}$$

‘napačno’ seštevanje

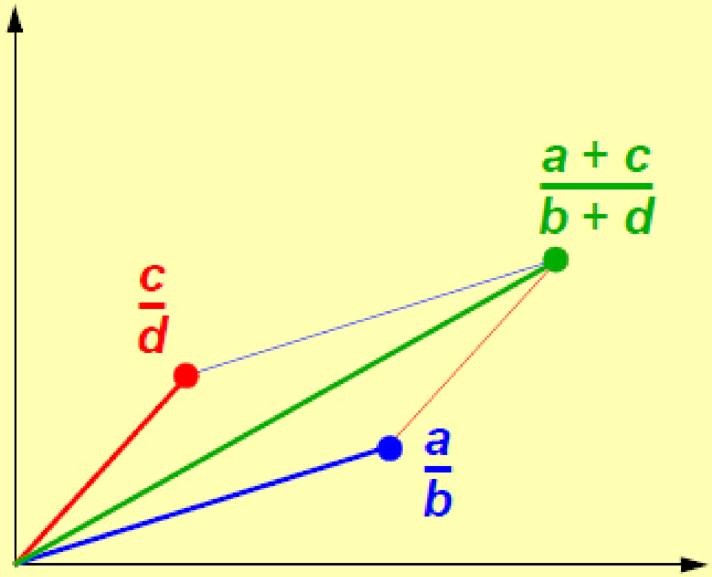


$\frac{7}{12}$

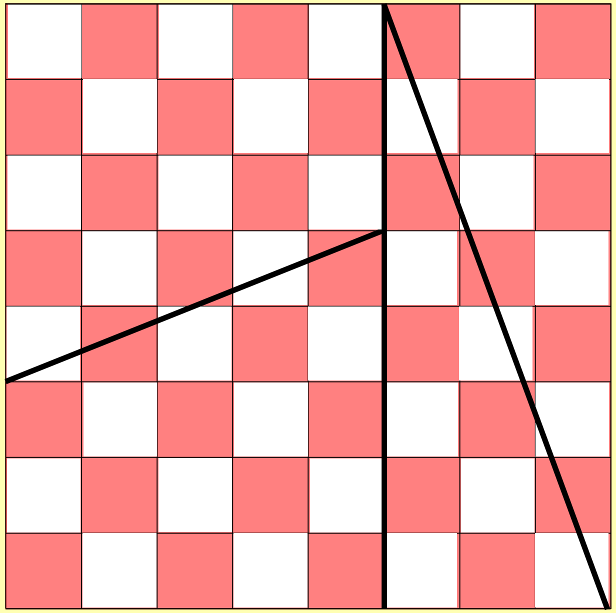
je med

$\frac{2}{5}$ in $\frac{5}{7}$

mediant števil $\frac{a}{b}$ in $\frac{c}{d}$ je med obema ulomkoma

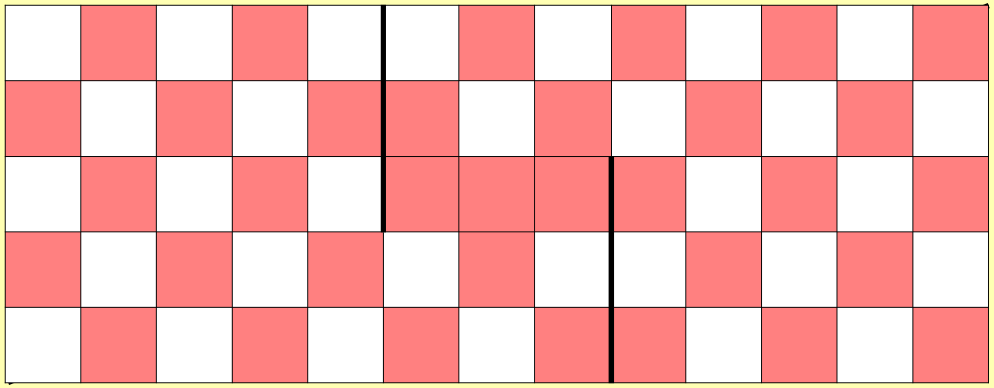


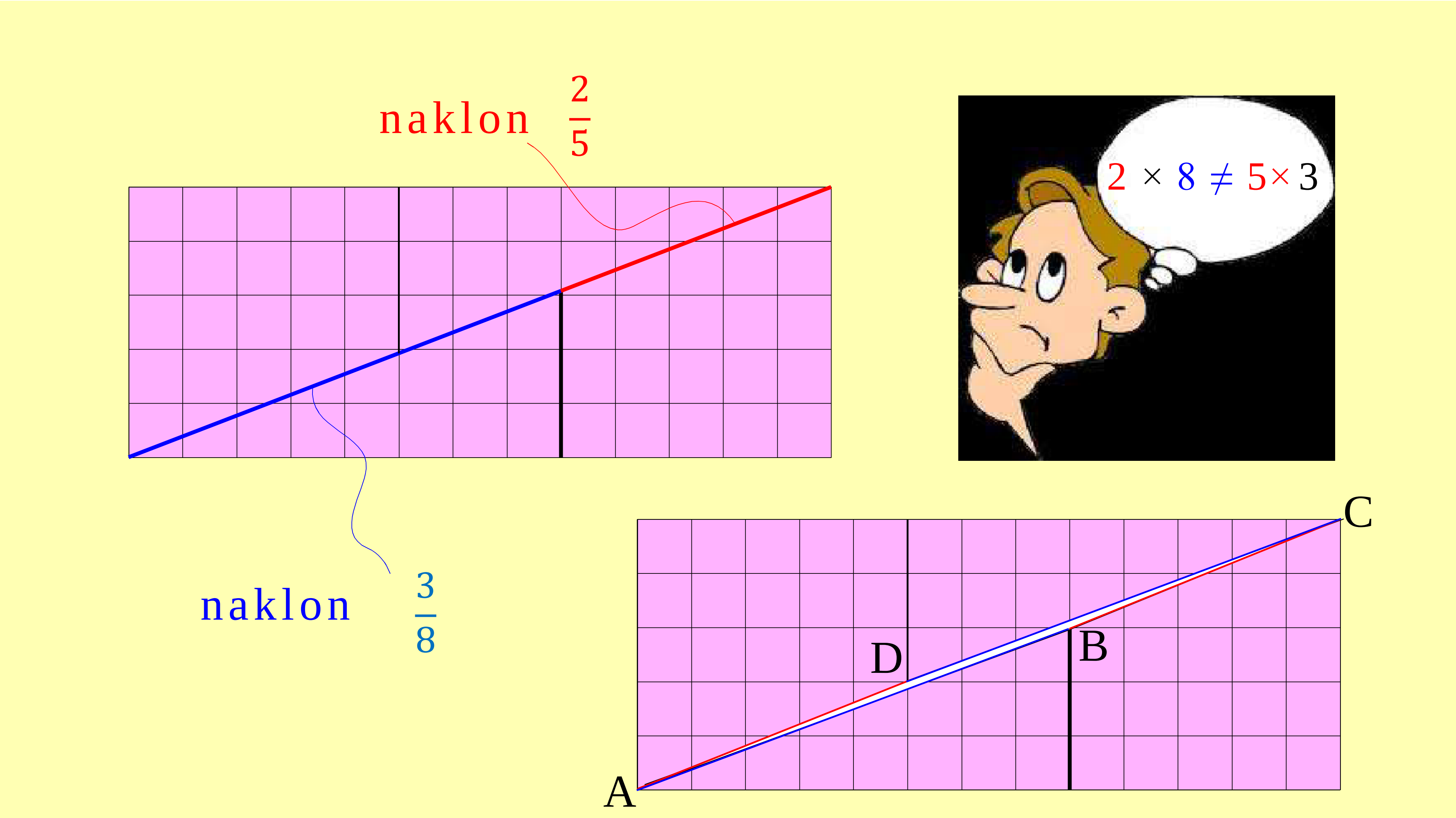
Opomba: „mediant“ števil $\frac{a}{b}$ in $\frac{c}{d}$ je število $\frac{a+c}{b+d}$



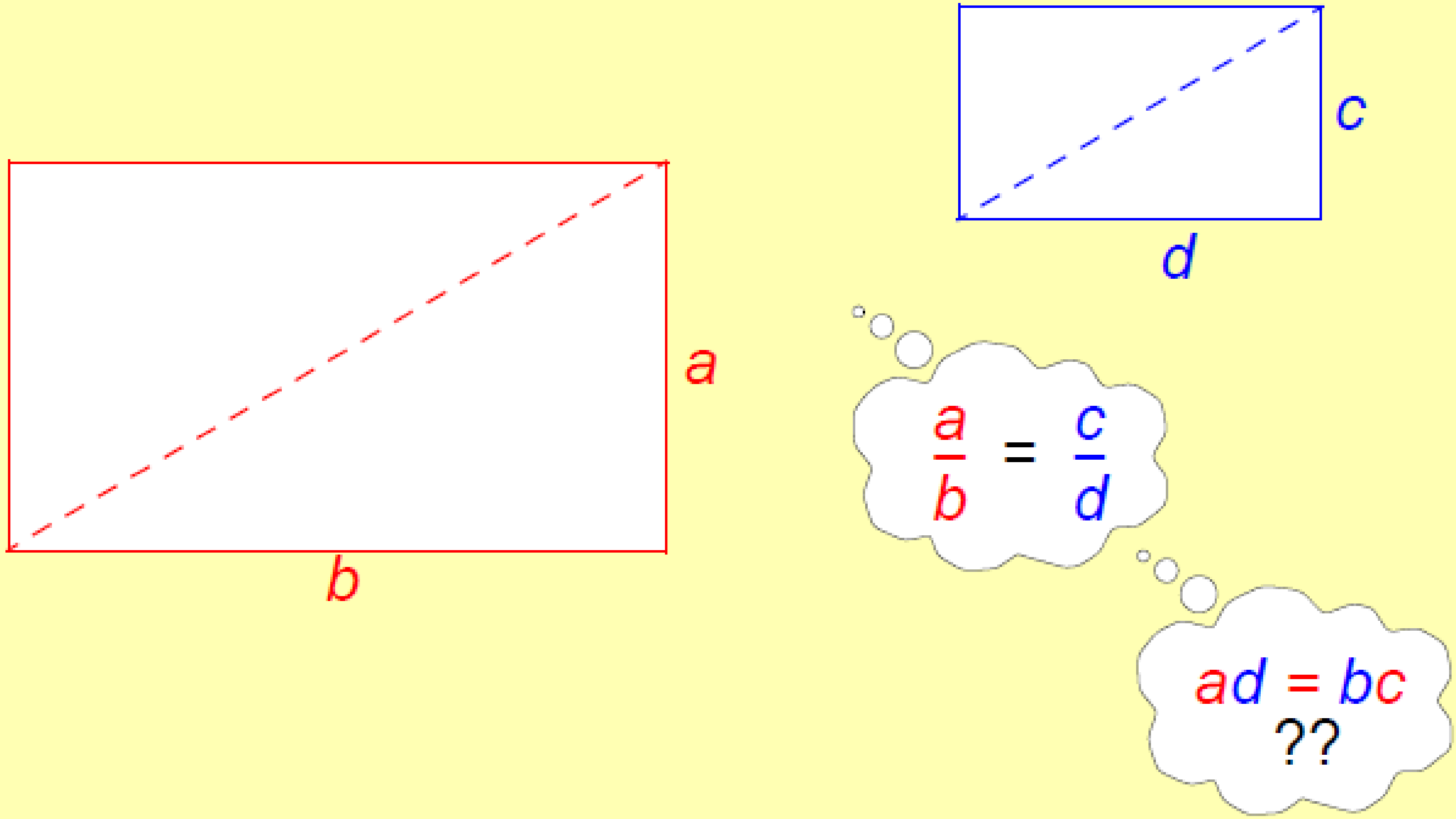
Lewis Carroll

64 = 65 ?

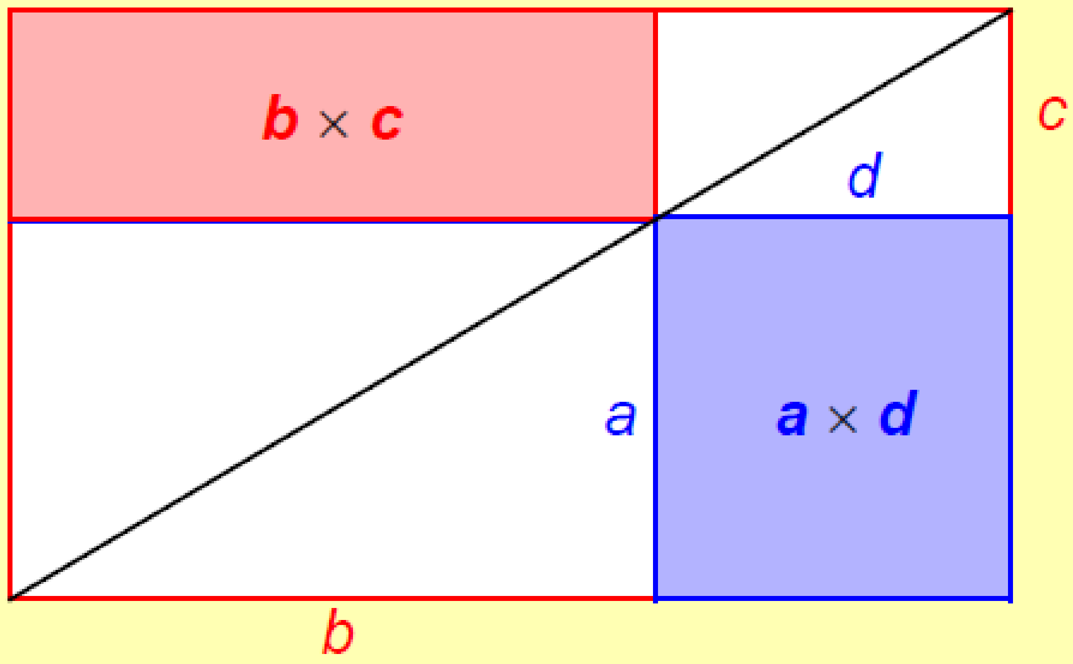



naklon 2/5
naklon 3/8
2 × 8 ≠ 5 × 3
A
B
C
D

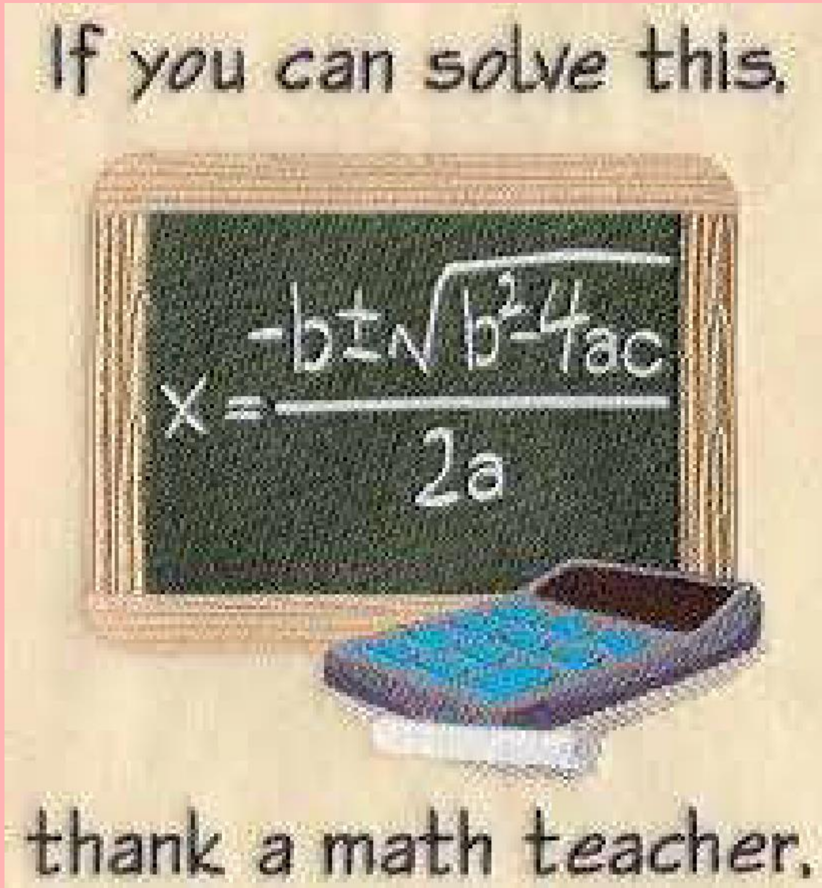
enaka naklona, enaka‘navzkrižna produkta’



enaka navzkrižna produkta

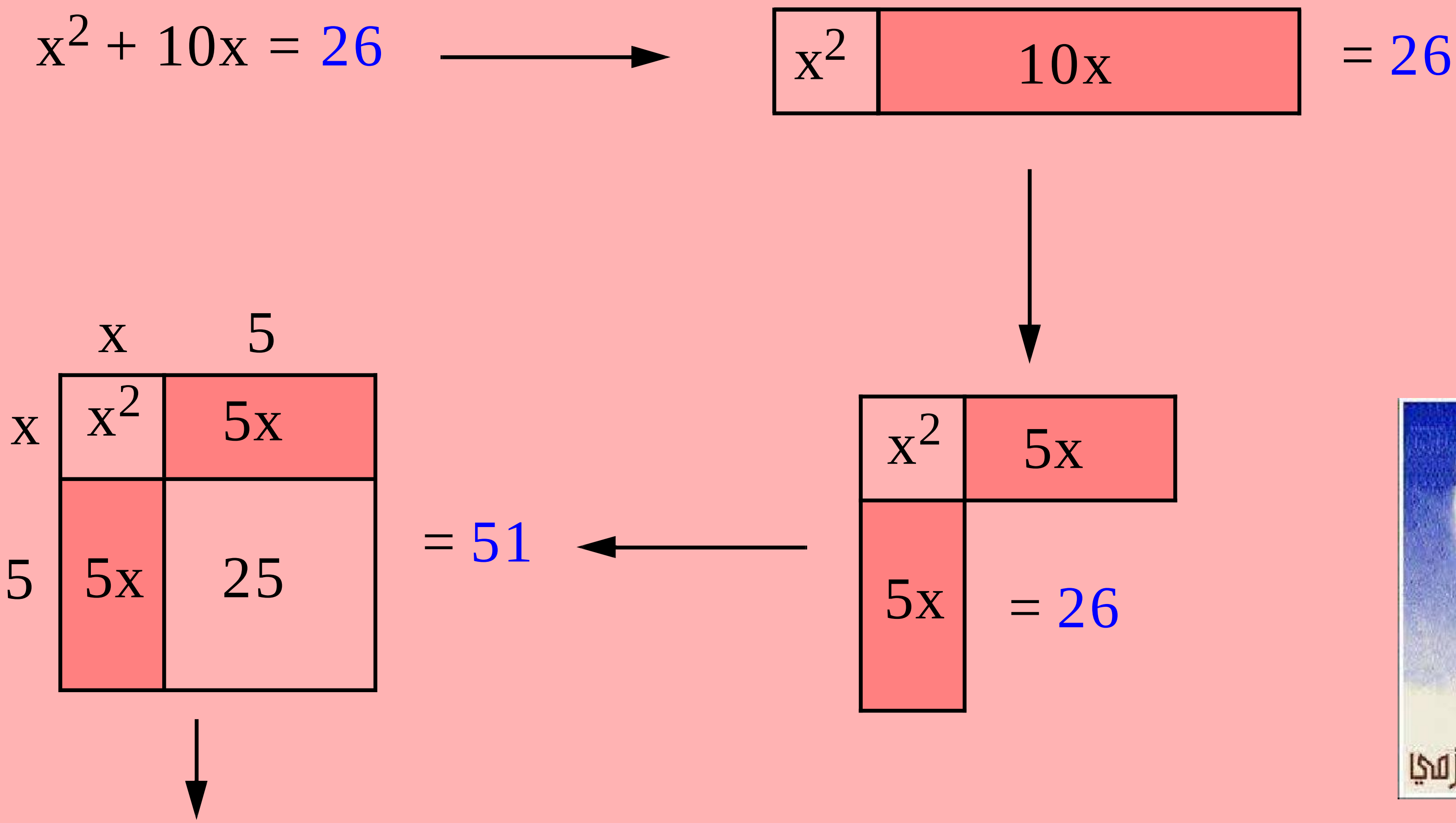


$$\frac{a}{b} = \frac{c}{d} \longleftrightarrow a \times d = b \times c$$



Če to znaš rešiti, se zahvali učitelju matematike.

$x^2 + 10x = 26$



$x = \sqrt{51} - 5$

zapuščina Al-Khwarizmija

Babilonci so algebrske izraze vnesli v geometrijo

Babilonska enačba

Imam 11 skladnih kvadratov. Dodam jim 7 pravokotnikov, ki imajo eno stranico enotsko, druga pa je enaka stranici kvadrata. Dobim ploščino 6,25. Koliko meri stranica kvadrata?

Babilonska enačba

Imam 11 skladnih kvadratov. Dodam jim 7 pravokotnikov, ki imajo eno stranico enotsko, druga pa je enaka stranici kvadrata. Dobim ploščino 6,25. Koliko meri stranica kvadrata?

Babilonska rešitev v današnjem jeziku matematike:

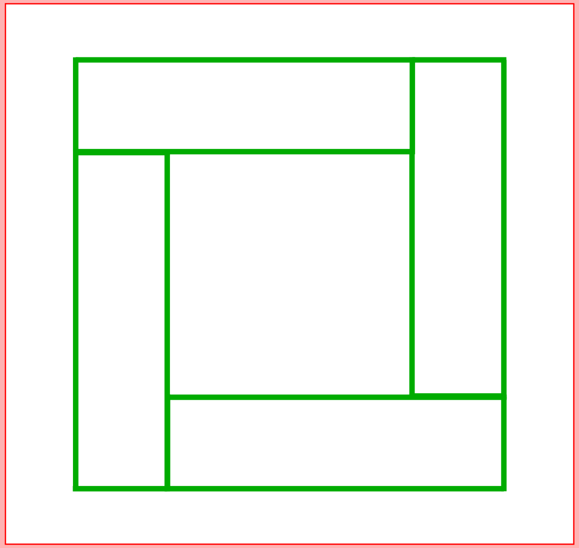
$11x^2 + 7x = 6.25$ → (! , x 11) → $11^2x^2 + 7.11x = 68.75$

$11x = u$

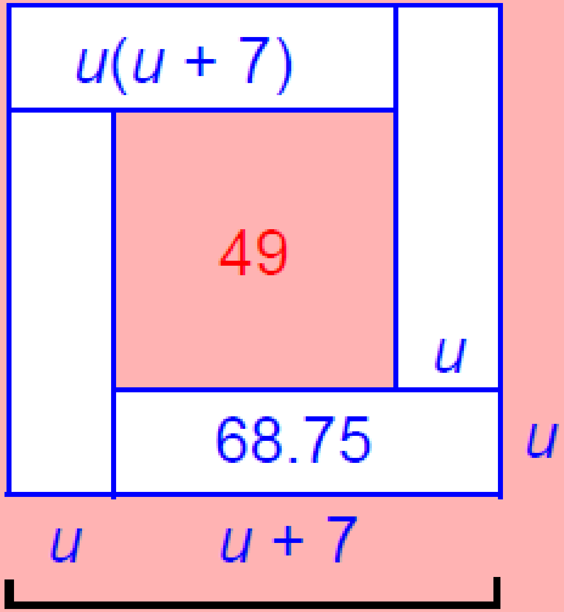
$u^2 + 7u = 68.75$ → $(u + 3.5)^2 = 68.75 + 12.25$ (= 9^2)

$u = 5.5$ → $x = 0.5$

oblikovanje: P. Kjaerholm

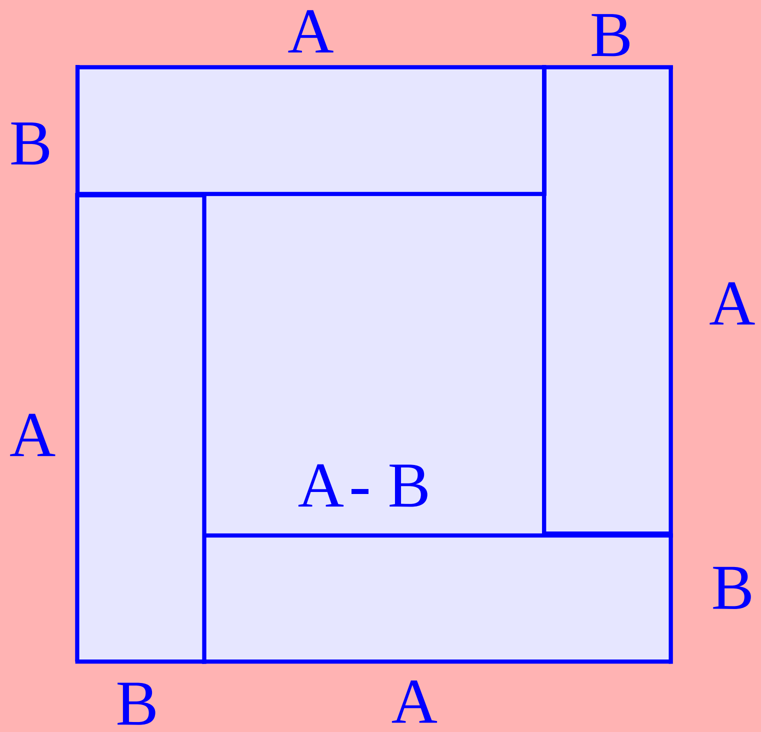


$$u^2 + 7u = 68.75 \text{ ali } u(u+7) = 68.75$$

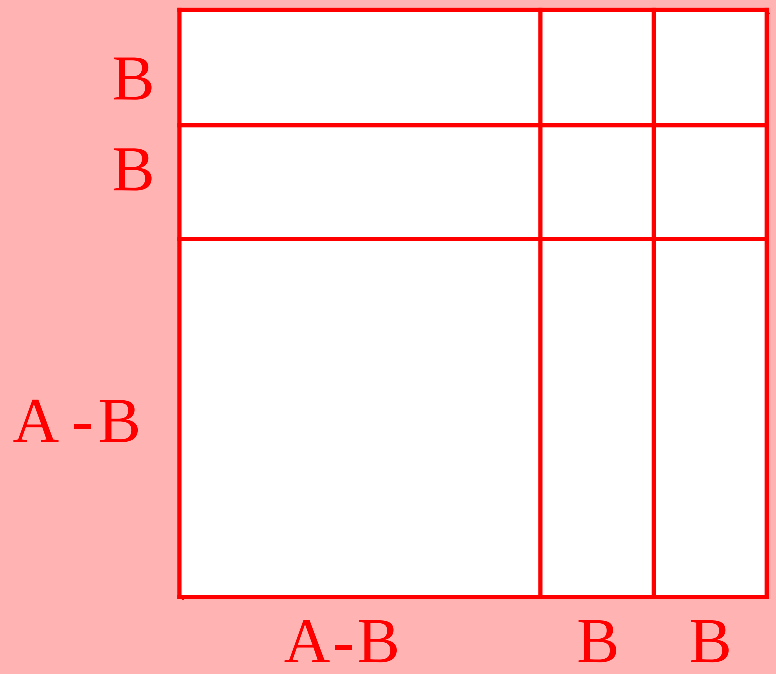


$$= 49 + 4 \cdot 68.75 = 324$$

$$2u + 7 = \sqrt{324} = 18 \longrightarrow \dots$$



$$(A+B)^2 = (A-B)^2 + 4AB$$



‘Evklidovi elementi’
(knjiga 2, poglavje 8)

$$(A + B)^2 = (A - B)^2 + 4AB$$

$$ax^2 + bx + c = 0 \longrightarrow x(ax + b) = -c$$

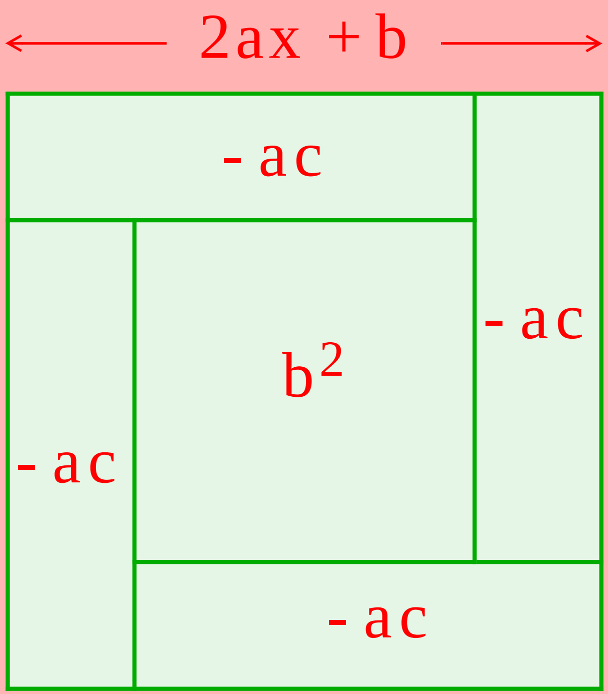
$$ax(ax + b) = -ac$$

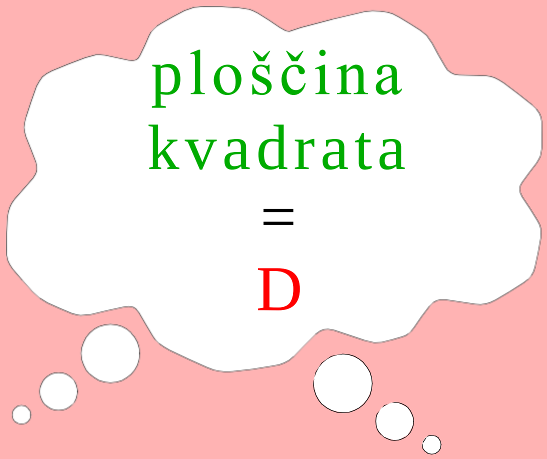
B A

$$(2ax + b)^2 = (b)^2 + 4(-ac)$$

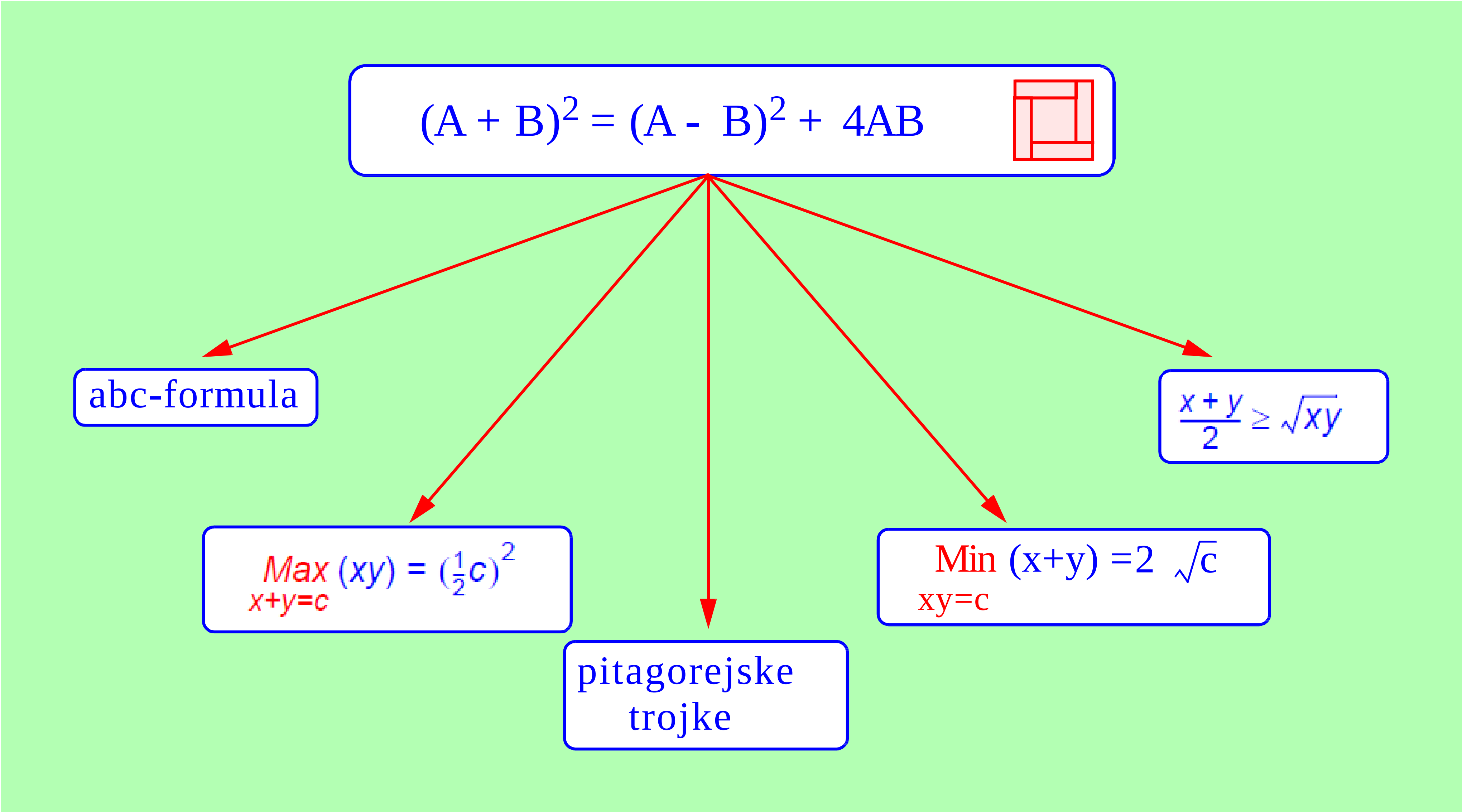
$a > 0$

$c < 0$





„diskriminantna miza“



$(A + B)^2 = (A - B)^2 + 4AB$
abc-formula
$\underset{x+y=c}{Max}(xy) = (\frac{1}{2}c)^2$
pitagorejske trojke
$\underset{xy=c}{Min}(x+y) = 2\sqrt{c}$
$\frac{x+y}{2} \geq \sqrt{xy}$

Nazaj k Fibonacciju

1 1 2 3 5 8 13 21 34 55 89 144 233 377 ...

naraščajoče zaporedje

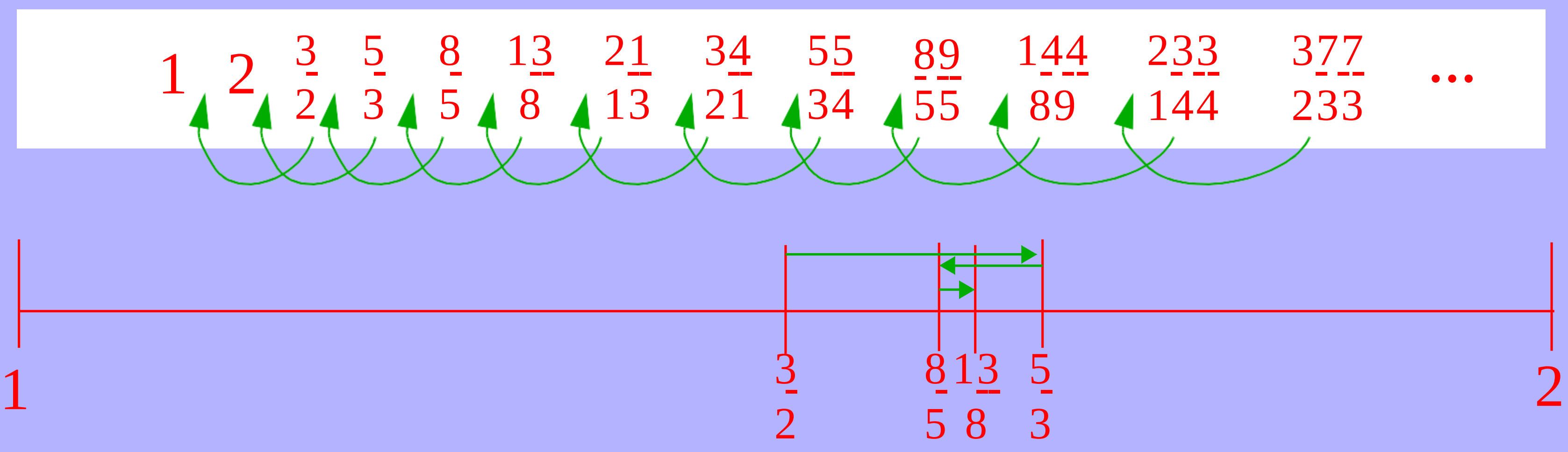
$$1 \quad 2 \quad \frac{3}{2} \quad \frac{5}{3} \quad \frac{8}{5} \quad \frac{13}{8} \quad \frac{21}{13} \quad \frac{34}{21} \quad \frac{55}{34} \quad \frac{89}{55} \quad \frac{144}{89} \quad \frac{233}{144} \quad \frac{377}{233} \quad \ldots$$

vsak ulomek je „mediant" dveh predhodnih členov

Nazaj k Fibonacciju

1 1 2 3 5 8 13 21 34 55 89 144 233 377 ...

naraščajoče zaporedje



1 1 2 3 5 8 13 21 34 55 89 144 233 377 ...

naraščajoče zaporedje

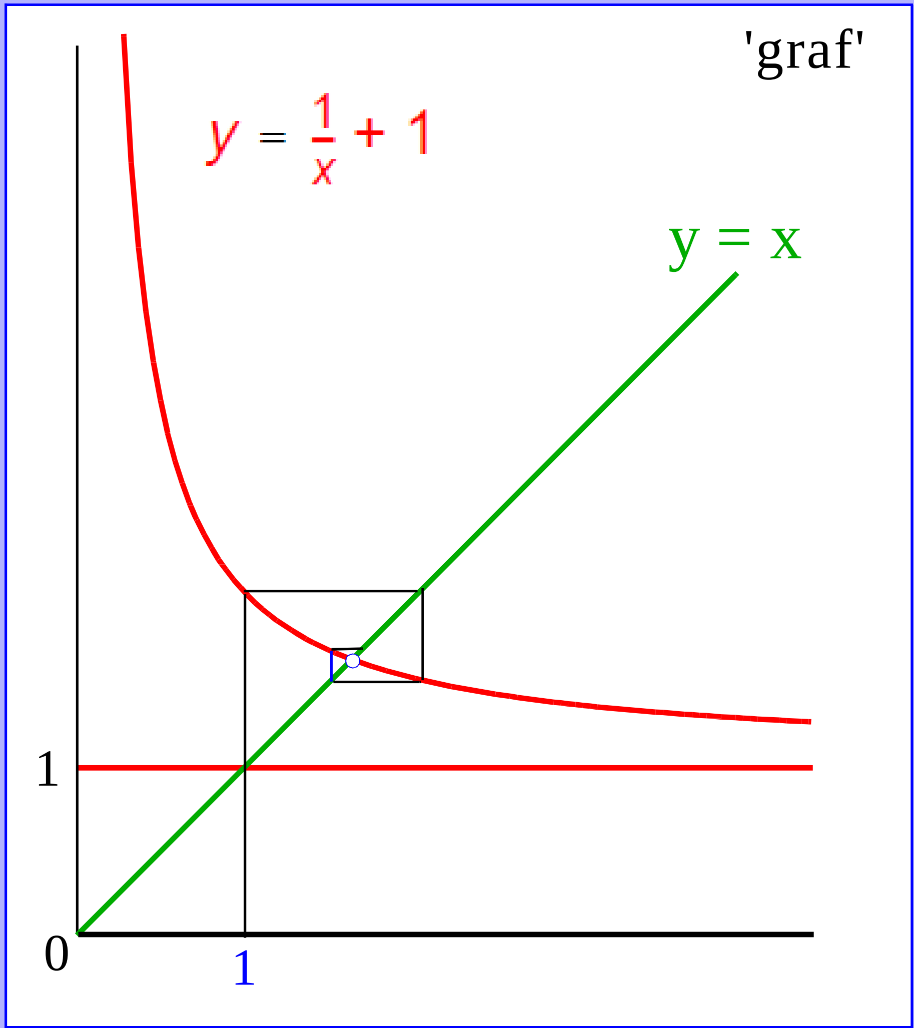
1	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$	$\frac{13}{8}$	$\frac{21}{13}$	$\frac{34}{21}$	$\frac{55}{34}$	$\frac{89}{55}$	$\frac{144}{89}$	$\frac{233}{144}$	$\frac{377}{233}$	...
1	2	1.5	1.6667	1.6	1.625	1.6154	1.6190	1.6176	1.6181	1.6180	1.6181	1.6180	

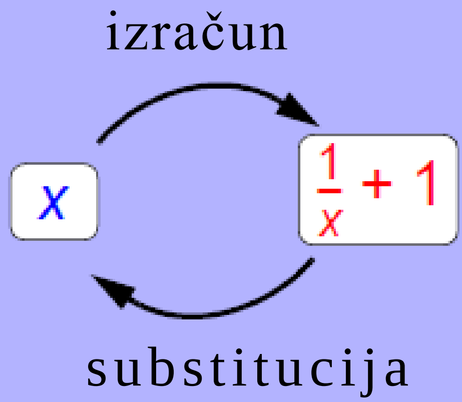
$a \qquad b \qquad a + b$

$$g = \frac{b}{a}$$

$$g^* = \frac{a+b}{b} = \frac{a}{b} + 1$$

$$g^* = 1 + \frac{1}{g}$$

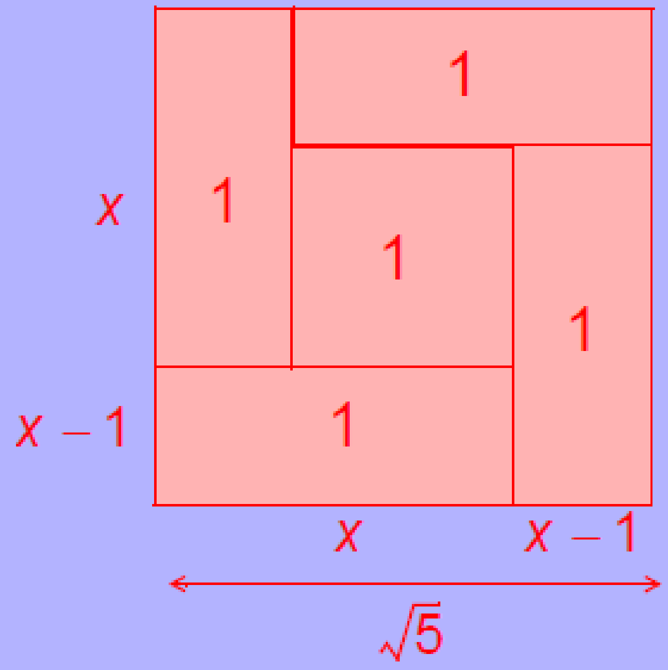

'graf'
y = 1/x + 1
y = x
1
0
1


izračun
x
1/x + 1
substitucija

$$x = \frac{1}{x} + 1$$

$$\Updownarrow$$

$$x(x-1) = 1$$

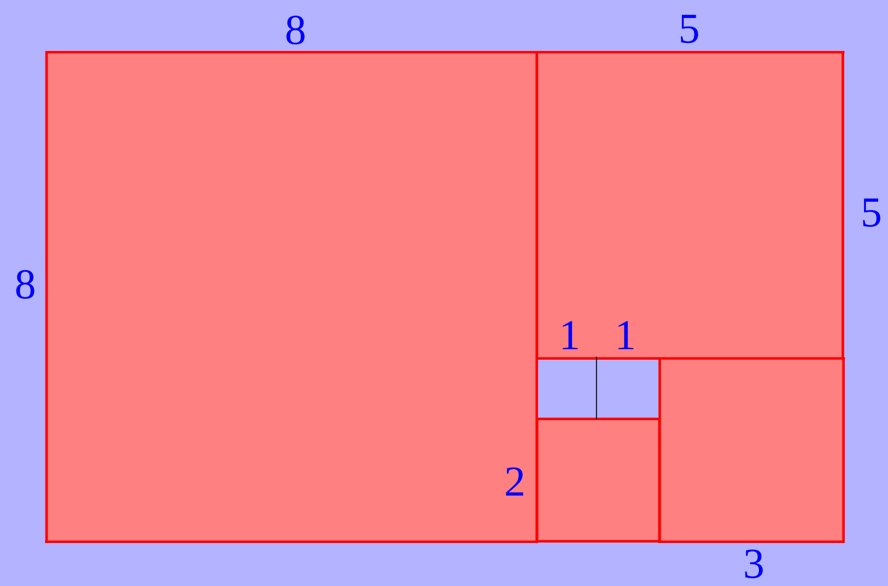


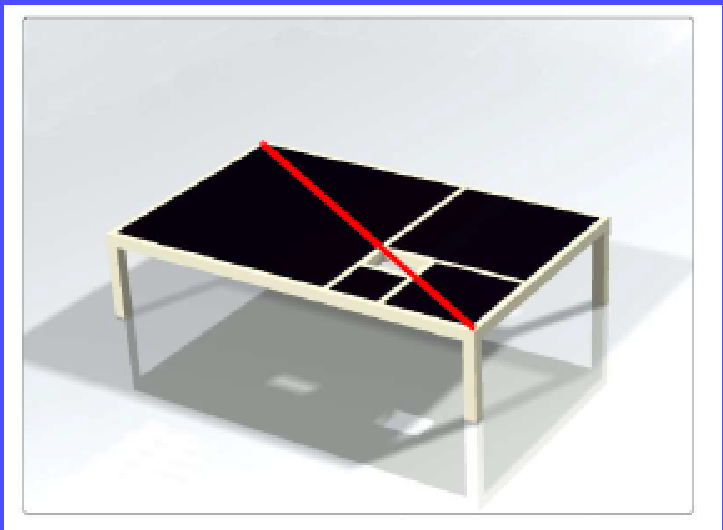
$$2x - 1 = \sqrt{5}$$

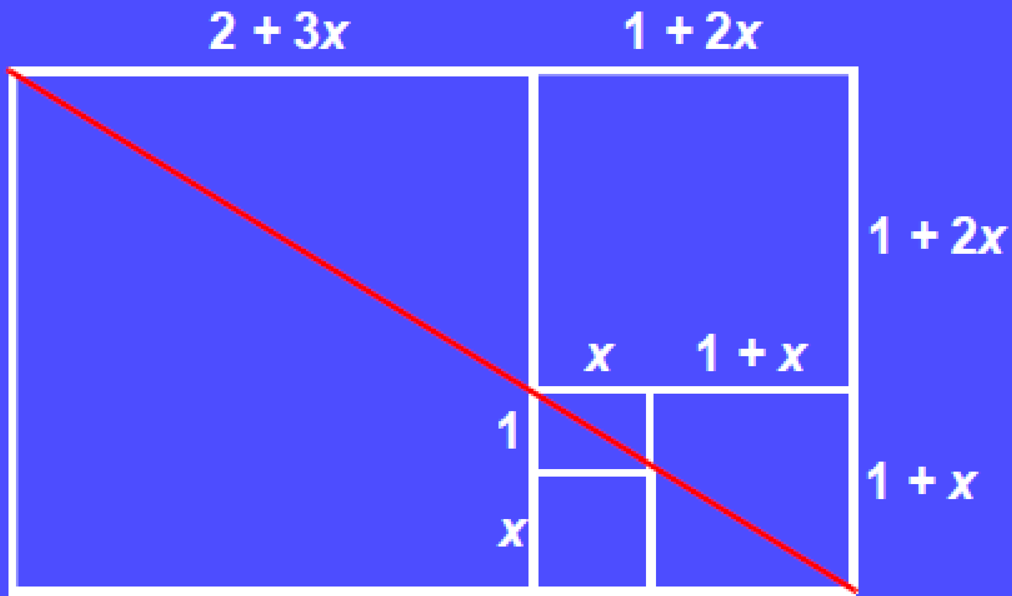
$$x = \frac{1}{2} + \frac{1}{2}\sqrt{5}$$

število zlatega reza


‘Fibonaccijeva miza’
Nizozemsko oblikovanje


8
5
5
8
1
1
2
3





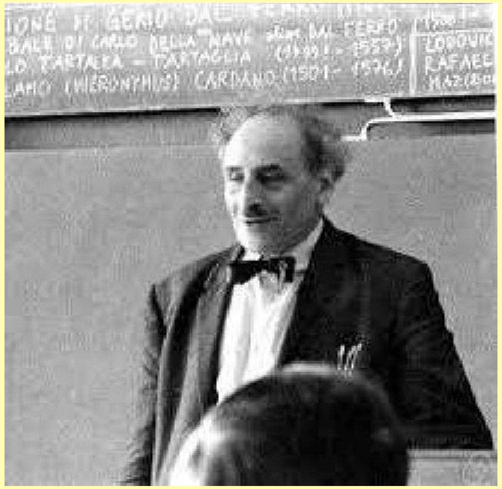
$$\frac{1}{x} = \frac{2+3x}{3+5x} \leftrightarrow 3 + 5x = 2x + 3x^2$$

$$\updownarrow$$

$$x(x-1) = 1$$

Veščine, ki se potrebne za uvid, lahko pridobivamo z vajo (načrtno, nenačrtno), kar je dobro.
Kaj pa je pomanjkljivost?
Rutinske naloge zakrijejo te veščine, ne vračamo se več nazaj v izhodišče, kar se običajno dogaja.
Kako med učenjem ohranjati vpogled v procese pridobivanja novega znanja? Kako vzpodbujati trajnost naučenega, s poudarkom na procesih shematizacije?

ICME 1980 Berkeley



Hans Freudenthal
1905 -1990

(Berkeley 1980, Eden od dvanajstih velikih problemov poučevanja matematike)

Del odgovora: Algebra z modnimi mizami

Hvala!