Dokazovanje od Močnika do računalnika

Zlatan Magajna

Pedagoška fakulteta, Univerza v Ljubljani

3. mednarodna konferenca
o učenju in poučevanju matematike
KUPM 2016

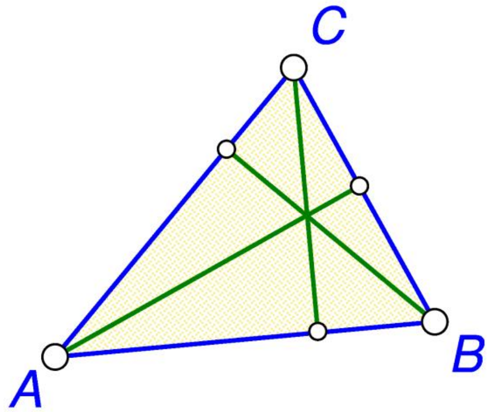
Zavod Republike Slovenije za šolstvo

REPUBLIKA SLOVENIJA
MINISTRSTVO ZA IZOBRAŽEVANJE, ZNANOST IN ŠPORT

EVROPSKA UNIJA
EVROPSKI SOCIALNI SKLAD
NALOŽBA V VAŠO PRIHODNOST

Naložbo sofinancira Evropski socialni sklad ter Ministrstvo za izobraževanje, znanost in šport, projekt Krepitev kompetenc strokovnih delavcev na področju vodenja inovativnega vzgojno-izobraževalnega zavoda v obdobju od 2016 do 2018

Zakaj dokazovati v šolski geometriji



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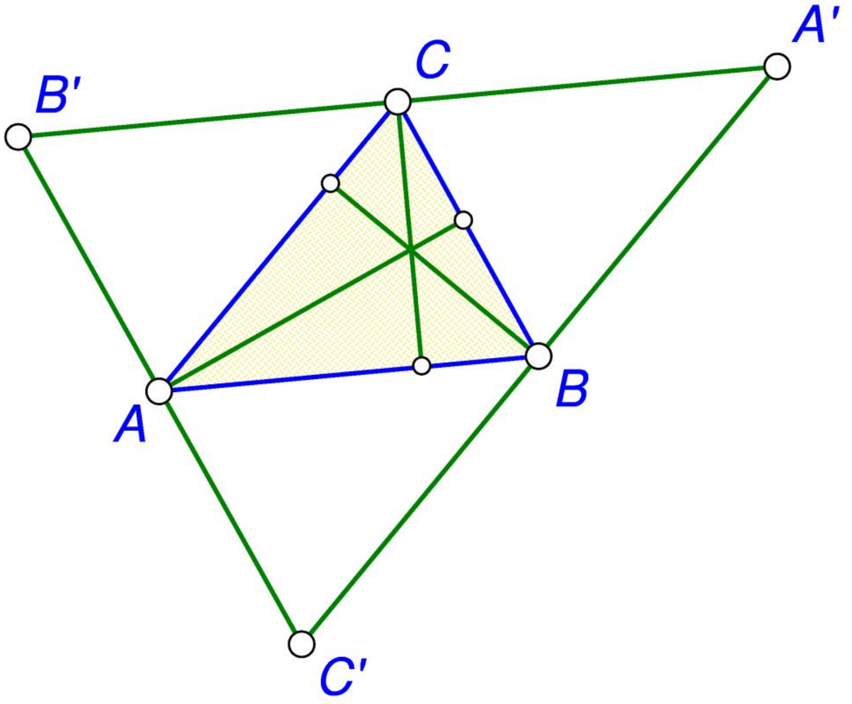

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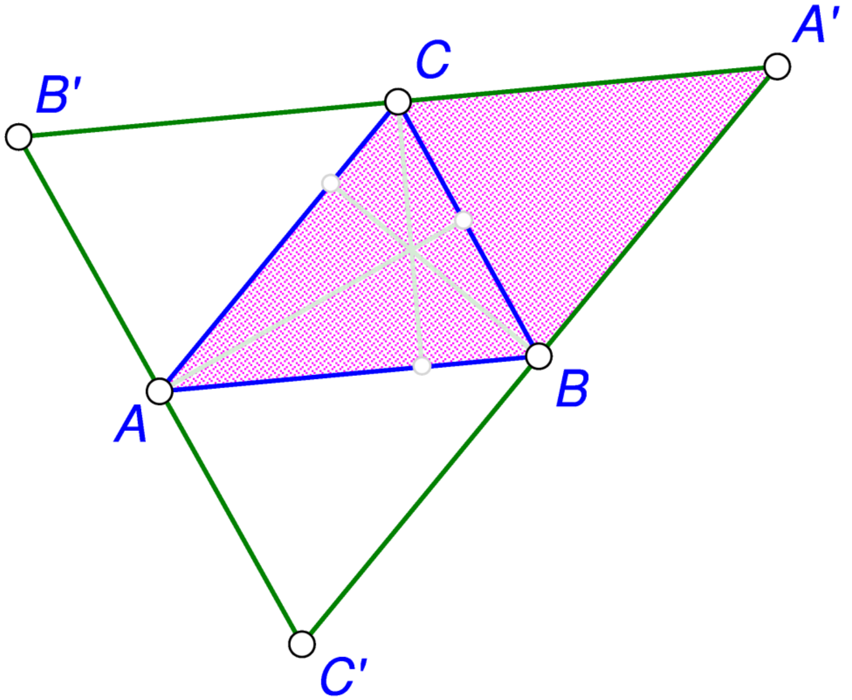

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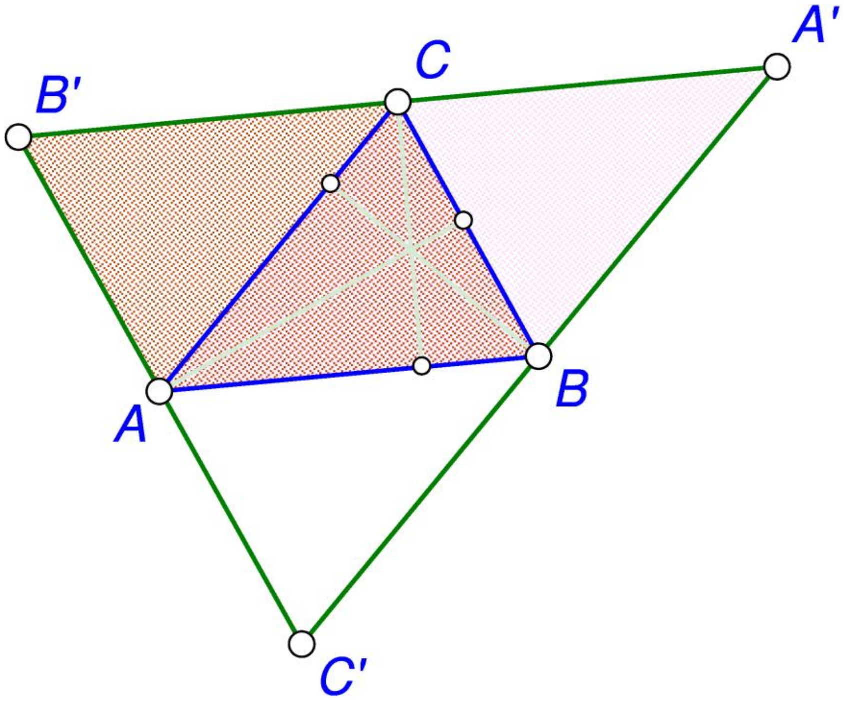
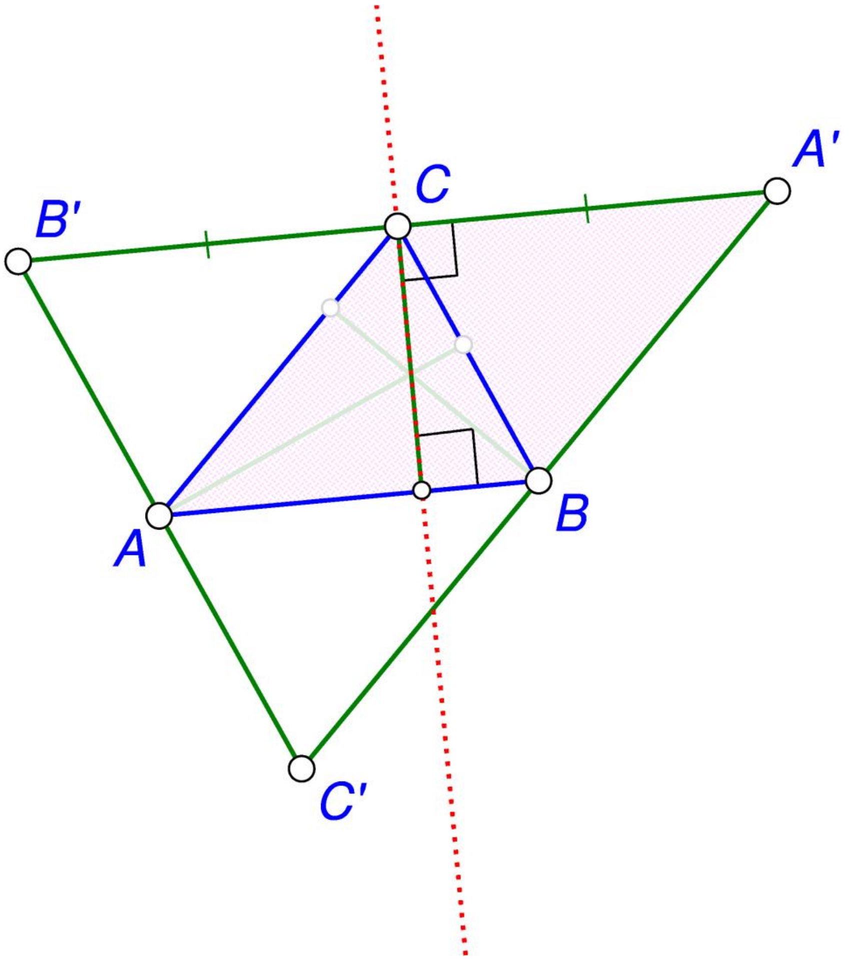


Zakaj dokazovati v šolski geometriji



Zakaj dokazovati v šolski geometriji

- V matematiki z dokazom verificiramo pravilnost trditve.
- V šolski matematiki ob dokazu predvsem razumemo, ZAKAJ neka trditev drži…

… a še mnogo več! (Hanna, 2000)

Dokazovanje v (ameriški) šolski geometriji

Herbst (2002)

- **Obdobje tekstov: Reproduciranje dokazov (1800-1850)**
- Obdobje originalov: Oblikovanje dokazov
- Obdobje nalog: Učenje dokazovanja

- Obdobje odkrivanja (eksperimentalni pristop h geometriji)
- Obdobje avtomatskega dokazovanja (dokazovanje z računalnikom)

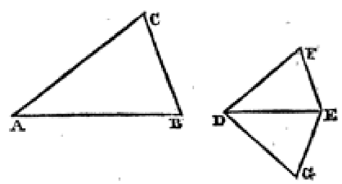
- "... obravnavana je **geometrija, kakršna je zapisana v (klasičnih) tekstih.**"
- Dokazi so zapisani kot tekst (paragraph form).
- **Znati geometrijo** = poznati izreke in **znati ponoviti dokaz izrekov**.

Šolska izdaja Evklidovih Elementov (1822)

PROP. V. THEOR.

If the sides of two triangles (ABC, DEF), about each of their angles, be proportional (AB to BC, as DE to EF; BC to AC, as EF to DF; and therefore, by ordinate equality, AB to AC, as DE to DF), the triangles are equiangular, having their equal angles opposite to the homologous sides.

At the extremes of any side DE, of either triangle, as DEF, make angles EDG and DEG equal to the angles A and B at the extremes of the side AB, which is homologous to DE; the remaining angle G of the triangle DEG, is equal to the remaining angle C of the triangle ABC (32. 1).



And, because the triangles ABC, DEG are equiangular, BA is to AC, as ED to DG (4. 6), and BA is to AC, as ED to DF (Hyp.), therefore ED is to DG, as ED to DF (11. 5), and so DG and DF are equal (9. 5); in like manner it may be proved, that EG and EF are equal, therefore the triangle DEG is equilateral to the triangle DEF, and of course equiangular to it [8. 1], and DEG is equiangular to ABC [Constr.], therefore the triangle ABC is equiangular to DEF, having the angle A equal to EDF, B to DEF, and C to F [Ax. 1. 1], namely, having those angles equal, which are opposite to the homologous sides.

PROP. VI. THEOR.

If two triangles (ABC, DEF, see fig. to preced. prop.) have an angle (A) of one, equal to an angle (EDF) of the other, and the sides about the equal angles proportional (BA to AC, as ED to DF); the triangles are equiangular, having those angles equal, which are opposite to homologous sides.

With either leg DE, of either of the equal angles A and EDF, and at either extreme of it D, make the angle EDG equal to A, and at E, the angle DEG equal to B; the remaining angle G of the triangle DEG, is equal to the remaining angle C of the triangle ABC (32. 1).

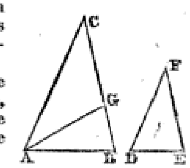
And, because the triangles ABC, DEG are equiangular, AB is to AC, as ED to DG (4. 6), but AB is to AC, as DE to DF [Hyp.], therefore DE is to DG, as DE is to DF (11. 5), and so DG and DF are equal (9. 5); and the angles EDG and EDF, being each of them equal to A (Constr. and Hyp.], are equal to each other (Ax. 1. 1), and DE is common to the two triangles EDG, EDF, therefore the triangle EDG is equiangular to the triangle EDF (4. 1); and the triangle ABC is equiangular to the triangle EDG (Constr.); therefore the triangle ABC is equiangular to DEF, having the angle B equal to DEF, and E to F (Ax. 1. 1), and therefore having those angles equal, which are opposite to the homologous sides.

PROP. VII. THEOR.

If two triangles (ABC, DEF), have an angle (C) of one, equal to an angle (F) of the other, and the sides about two of the other angles proportional (BA to AC, as ED to DF), and the two remaining angles (B and E) either both less, or both not less than a right angle; the triangles are equiangular, having the angles equal, about which are the proportional sides.

Let first the angles B and E be both less than a right angle. The triangles ABC, DEF are equiangular, the angles CAB and FDE being equal.

For, if the angles CAB and FDE be not equal, let one of them, if possible, as CAB, be the greater, and at the point A, with the right line CA, make the angle CAG equal to D (23. 1).



Because, in the triangles CAG, FDE, the angles C and F are equal [Hyp.], and the angles CAG and D also equal [Constr.], the remaining angles AGC and E are equal [32. 1]; therefore these triangles are equiangular, and of course CA is to AG, as FD to DE [4. 6]: and CA is to AB, as FD to DE [Hyp.], therefore CA is to AG, as CA to AB (11. 5), and so AG and AB are equal [9. 5]; therefore the angles AGB and ABG are equal [5. 1], and therefore both acute [Cor. 17. 1]; and because AGB is acute, AGC is obtuse [13. 1], and therefore the angle E, equal to AGC, is obtuse, which is absurd, the

https://books.google.si/books?id=tncAAAAMAAJ&pg=PA485&lpg=PA485&dq=euclid+elements&source=bl&ots=6FN3gVLyKQ&sig=IKcudgW9S4IruRIN0wYMp8OKIgk&hl=sl&sa=X&ei=jmDqVMjiLIOzPdi5gdgB&ved=0CDUQ6AEwAzh4#v=onepage&q=euclid%20elements&f=true

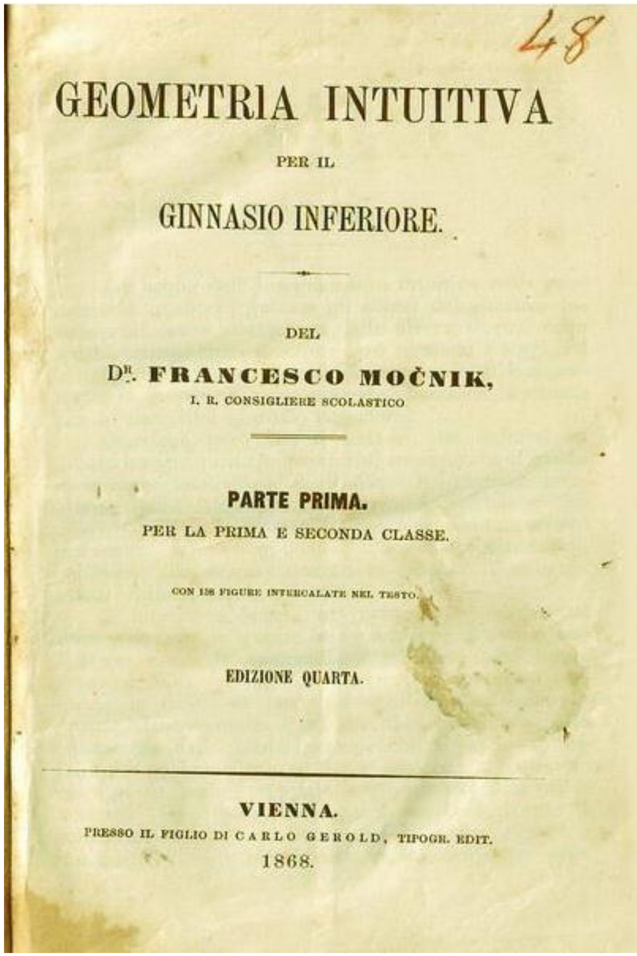
Dokazovanje v (ameriški) šolski geometriji

Herbst (2002)

- Obdobje tekstov: Reproduciranje dokazov
- **Obdobje originalov: Oblikovanje dokazov (1850-1900)**
- Obdobje nalog: Učenje dokazovanja
- Obdobje odkrivanja (eksperimentalni pristop h geometriji)
- Obdobje avtomatskega dokazovanja (dokazovanje z računalnikom)

- Učbeniki **več ponazoritev** za boljše razumevanje in **hipotetične konstrukcije**.
- Avtorji izdelujejo **enostavnejše, razumljivejše** dokaze, zavestno preskočijo komplicirane posebne možnosti, **lažje dele dokazov prepustijo** samostojnemu delu študentom.

Franc Močnik



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GEOMETRIA INTUITIVA

PER IL

GINNASIO INFERIORE.

DEL

Dr. FRANCESCO MOČNIK,

I. R. CONSIGLIERE SCOLASTICO

PARTE PRIMA.

PER LA PRIMA E SECONDA CLASSE.

CON 158 FIGURE INTERCALATE NEL TESTO.

EDIZIONE QUARTA.

VIENNA.

PRESSO IL FIGLIO DI CARLO GEROLD, TIPOGR. EDIT.

1868.

Močnikova načela - geometrija

Nižja gimnazija

- ‘Na intuiciji temelječe učenje geometrije’
- Opazovalna geometrija, sistematično spoznavanje dejstev

Učiteljišče:

- ‘Strogo znanstveni pristop’
- Aksiomatska, deduktivna geometrija

Namen na intuiciji temelječega učenja geometrije je natančno spoznavanje različnih oblik v prostoru, kot tudi z njimi povezanih odnosov in zakonitosti, in tako na preprost in naraven način pripraviti učence na strogo znanstveno geometrijo v višji gimnaziji.

Močnikova načela – geometrija v nižji gimnaziji

- Predstavitev posameznih oblik, postopkov
- Ozaveščanje opazovanih lastnosti oblik
- **Izpeljava manj očitnih dejstev (izrekov) in postopkov**
- Neposredna uporaba spoznanih dejstev
- Reševanje praktičnih nalog (ko gre za količine)
- Natančno prerisovanje s table, izdelava čistopisa, prostoročno risanje

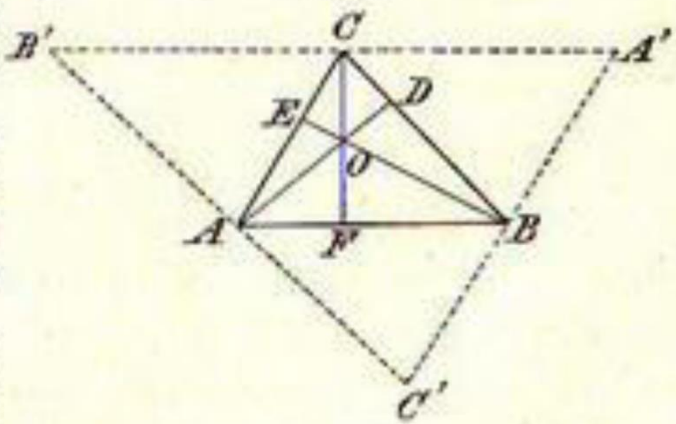
- Nad daljico načrtamo dva enakokraka trikotnika. Deduktivna utemeljitev, da zveznica vrhov razpolavlja kota ob vrhih trikotnikov in izhodiščno daljico.

Preprost primer dokaza

3. *Le tre altezze di un triangolo s' incontrano nello stesso punto.*

Nel triangolo ABC (Fig. 43) sia $AD \perp BC$, $BE \perp AC$ e $CF \perp AB$. Tirando per A, B, C le parallele a BC, AC ed AB si ottiene il triangolo $A'B'C'$ in cui A, B, C sono i centri dei lati, e AD, BE, CF sono le tre altezze del triangolo, rispettivamente perpendicolari ai lati del medesimo. Quindi secondo 1, anche AD, BE e CF devono tagliarsi in un punto.

Fig. 43.



Dokazovanje v (ameriški) šolski geometriji

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- Obdobje odkrivanja (eksperimentalni pristop h geometriji)
- Obdobje avtomatskega dokazovanja (dokazovanje z računalnikom)

- Učbeniki obravnavajo **metode in strategije dokazovanja**.
- Učbeniki vsebujejo **dokazovalne naloge**, katerih namen je učenje dokazovanja (ne pa učenje novih postopkov ipd.)
- Namen dokazovanja je, bolj kot verifikacija, **razvijanje mišljenja**.
- Razvit je bila zapis dokaza z **metodo dveh kolon**.

Obdobje odkrivanja (1960 - ?)

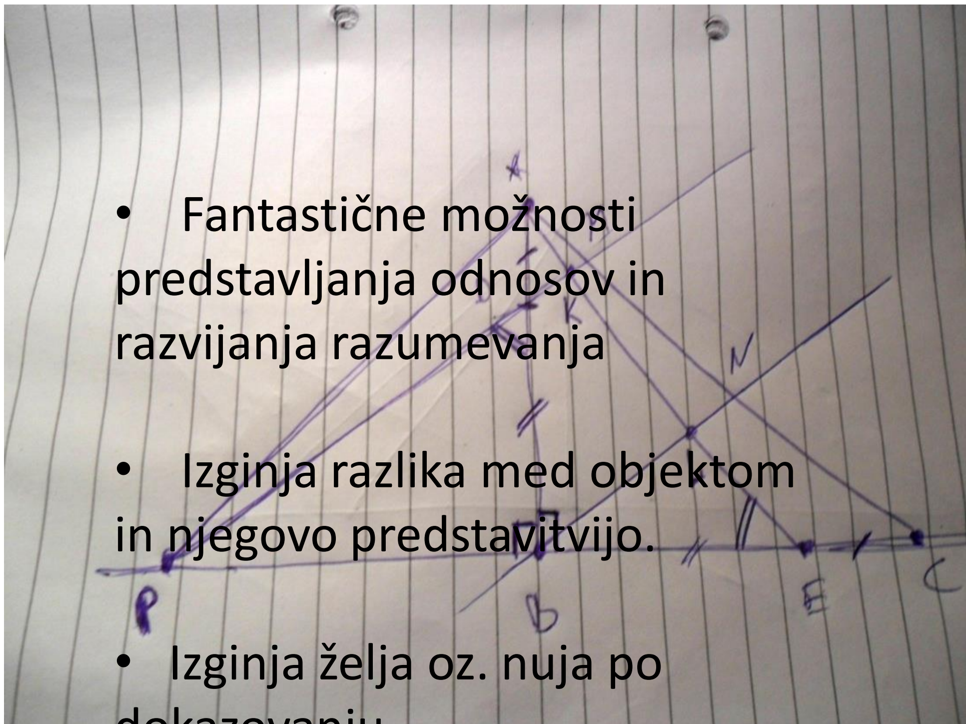
Na kurikularni ravni se **krepi**

- razumevanje pojmov
- predstavitve pojmov, odnosov
- odkrivanje značilnosti geometrijskih objektov
- povezovanje z drugimi vejami matematike (analitična geometrija, vektorji, trigonometrija…)

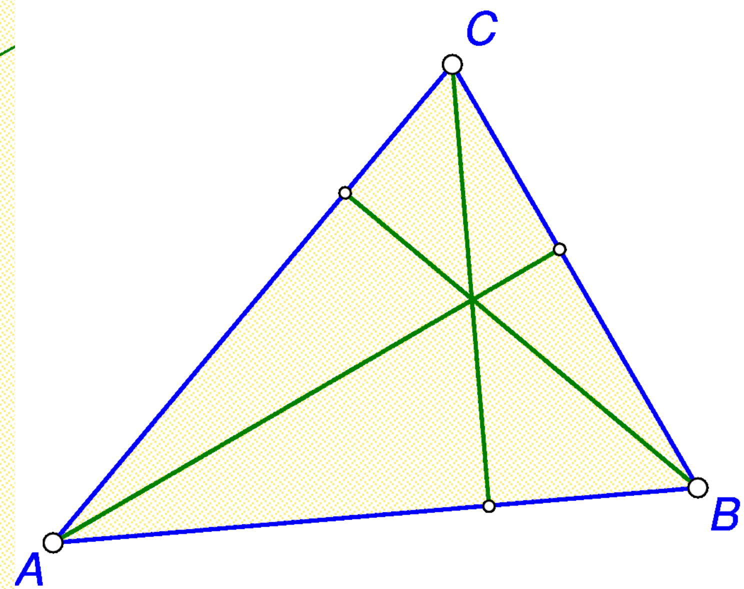
Na kurikularni ravni **bledi**

- aksiomatski pristop
- formalno dokazovanje
- deduktivno sklepanje
- zahtevnejše konstruiranje

Obdobje odkrivanja – dinamična geometrija

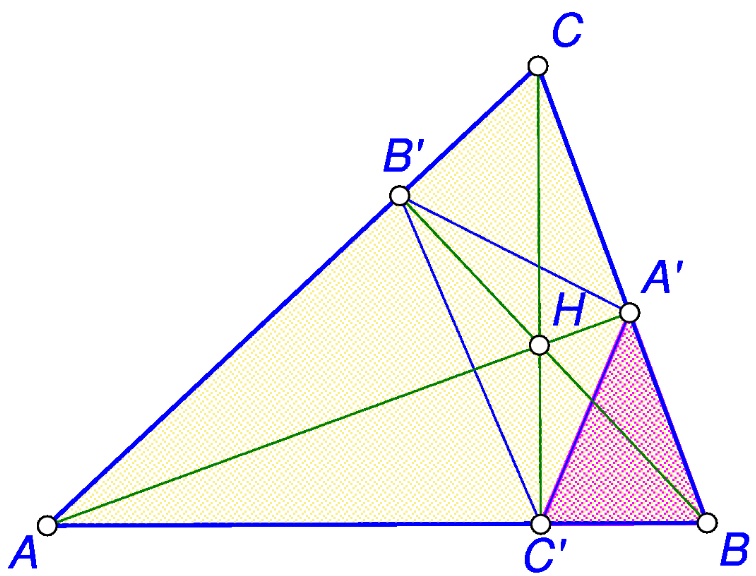


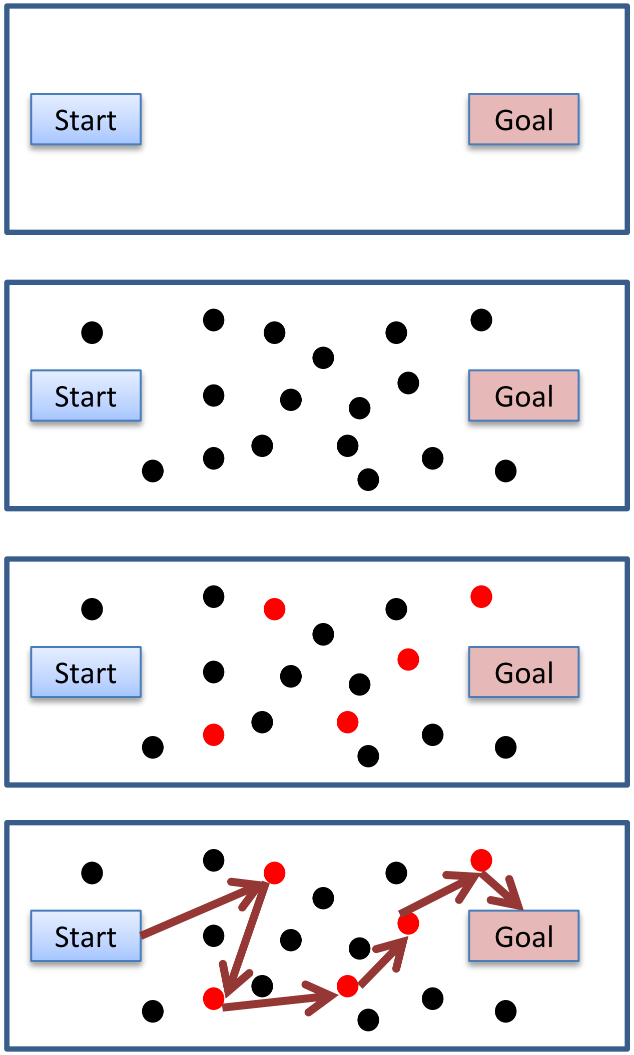
- Fantastične možnosti predstavljanja odnosov in razvijanja razumevanja
- Izginja razlika med objektom in njegovo predstavitvijo.
- Izginja želja oz. nuja po dokazovanju.



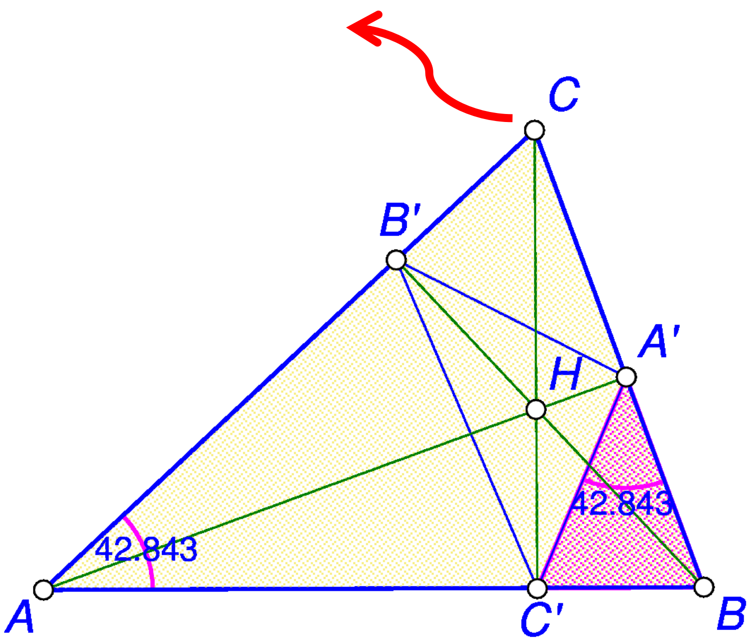
Nekatere ‘skice’ niso skice, temveč okna v Platonova nebesa. (Brown, 1999)

Dinamična geometrija in dokazovanje

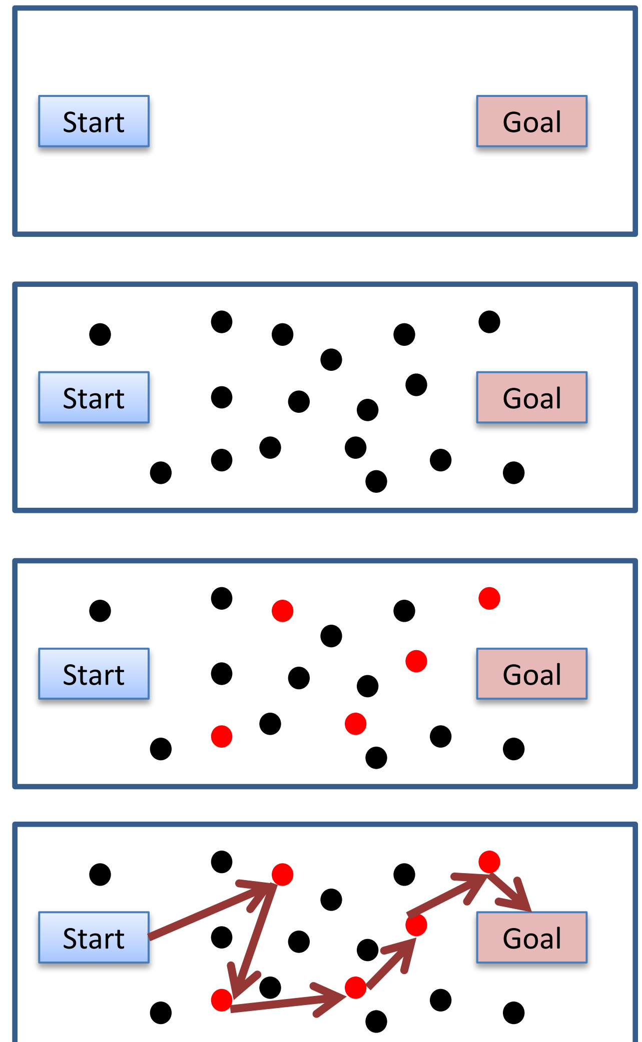




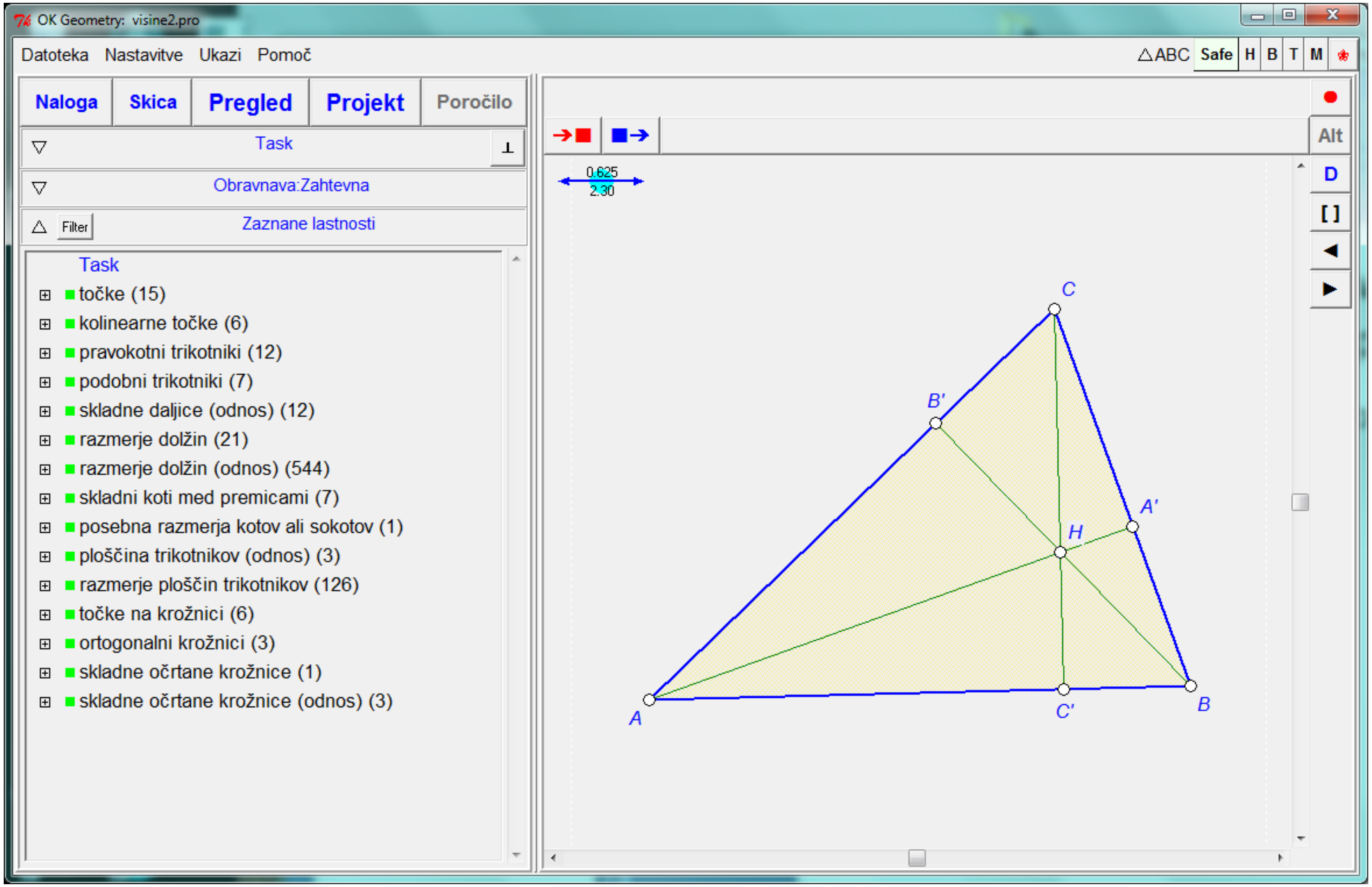
Dinamična geometrija in dokazovanje



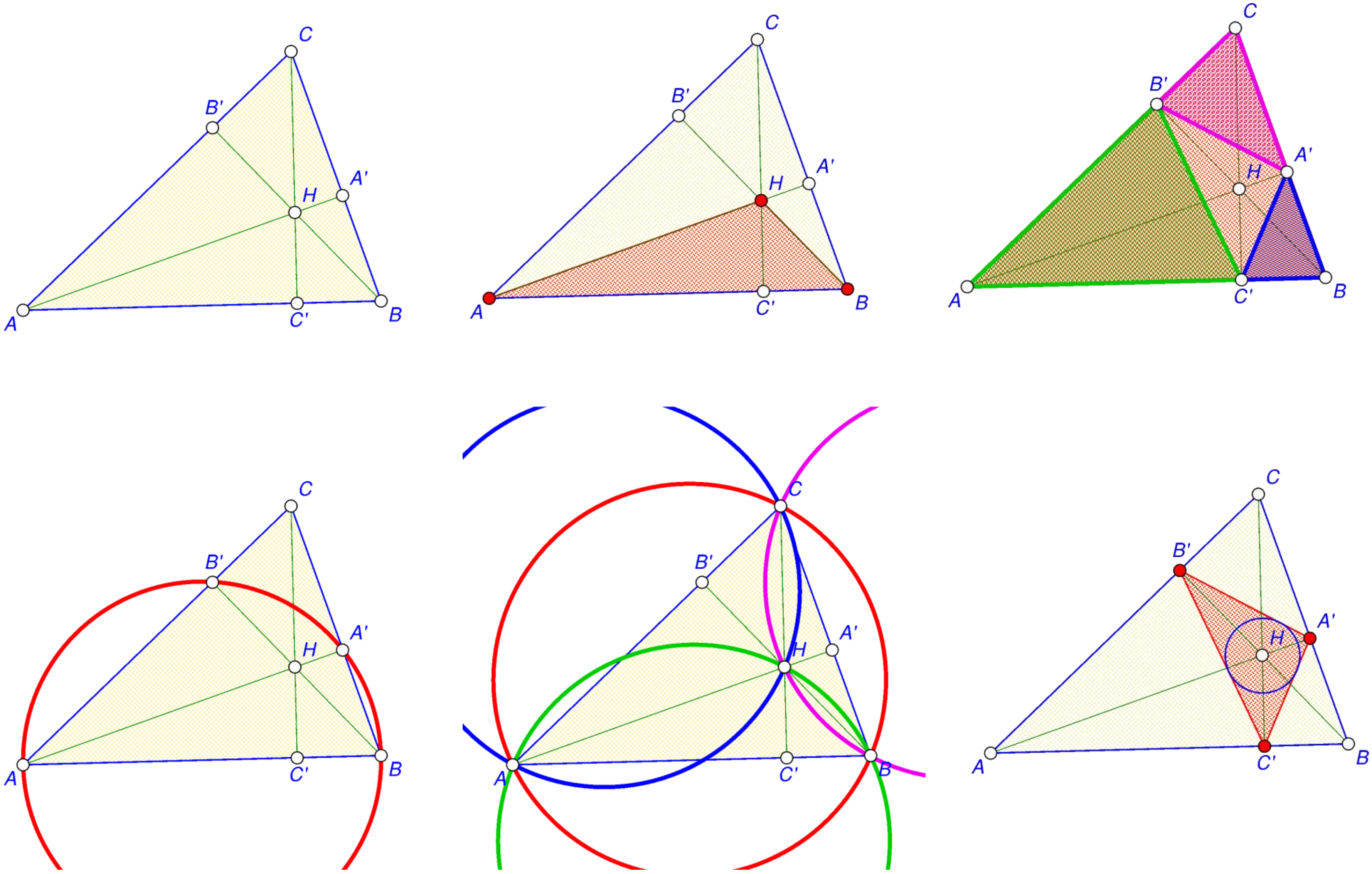
(Villiers, 2010; Laborde, 2000)



Avtomatsko opazovanje – OK Geometry



Avtomatsko opazovanje – OK Geometry



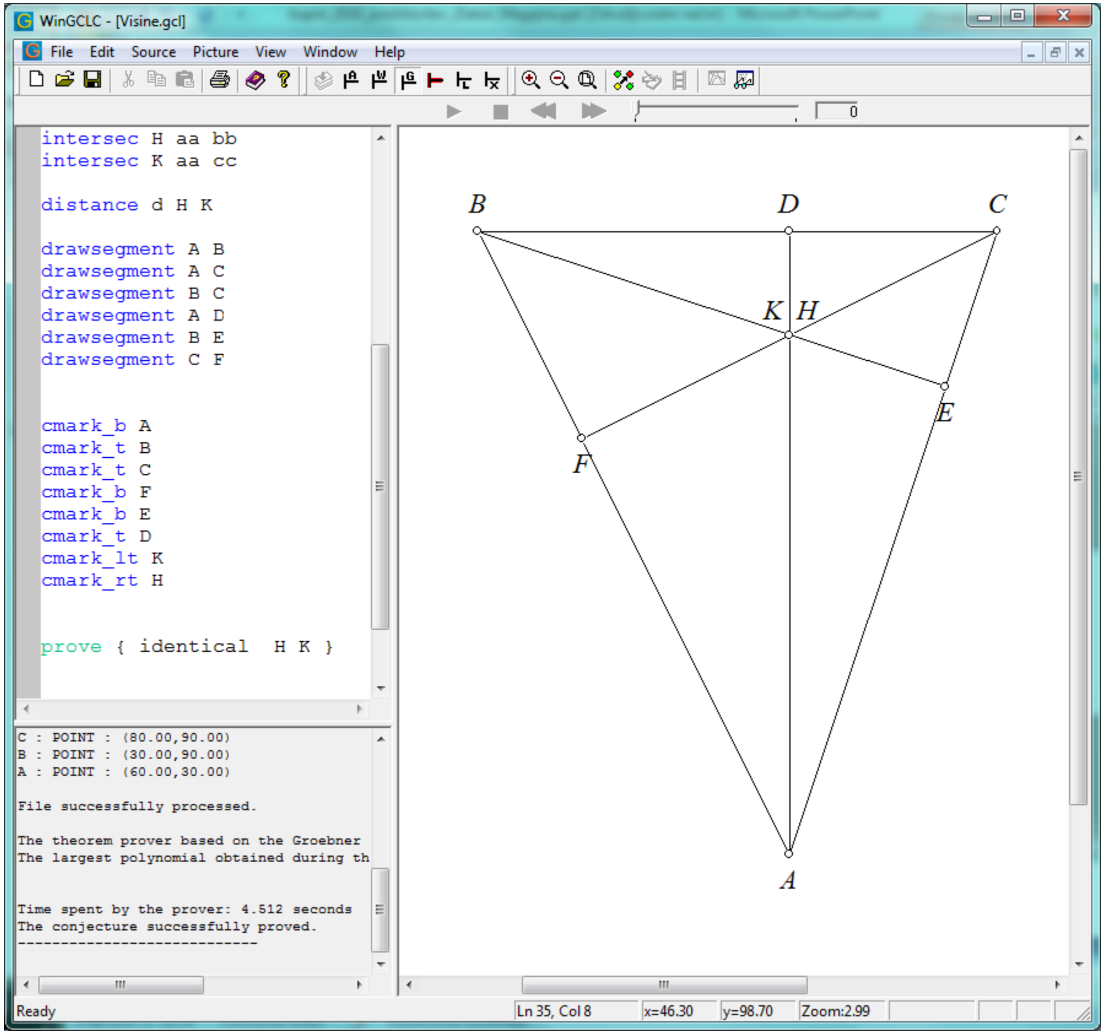
Obdobje avtomatskega dokazovanja (? - ?)

GCLC – Theorem prover (Janičić)

File successfully processed.

The theorem prover based on the Groebner bases method used. The largest polynomial obtained during the proof process contains 630 terms.

Time spent by the prover: 4.512 seconds
The conjecture successfully proved.



$$p_0 = (u_2 - u_1)x_2 + u_3x_1$$
$$p_1 = -u_3x_2 + (u_2 - u_1)x_1 + u_3u_1$$
$$p_2 = u_2x_4 + u_3x_3 - u_2u_1$$
$$p_3 = -u_3x_4 + u_2x_3$$
$$p_4 = -x_8x_1 + x_7x_2$$
$$p_5 = -x_8x_3 + x_7x_4 - u_1x_7 + u_1x_3$$
$$p_6 = x_9x_2 - u_2x_1$$
$$p_7 = (-u_3^2 - u_2^2 + 2u_2u_1 - u_1^2)x_1 + (-u_3u_2u_1 + u_3u_1^2)$$
$$p_8 = u_3x_9x_1 + (u_2^2 - u_2u_1)x_1$$
$$p_9 = (u_2 - u_1)x_9x_1 + u_3u_1x_9 - u_3u_2x_1$$
$$p_{10} = (-u_3^2 - u_2^2)x_3 + u_3u_2u_1$$
$$p_{11} = u_3u_1x_7x_3x_1 + (u_2^2u_1 - u_2u_1^2)x_3x_1$$
$$\begin{aligned}p_{12} = {} & (u_3^3u_2^2u_1 - 2u_3^3u_2u_1^2 + u_3^3u_1^3 + u_3u_2^4u_1 - 4u_3u_2^3u_1^2+ \\ & 6u_3u_2^2u_1^3 - 4u_3u_2u_1^4 + u_3u_1^5)x_8+ \\ & (u_3^4u_2u_1 - u_3^4u_1^2 + u_3^2u_2^3u_1 - 3u_3^2u_2^2u_1^2 + 3u_3^2u_2u_1^3- \\ & u_3^2u_1^4)x_7\end{aligned}$$
$$\begin{aligned}p_{13} = {} & (-u_3^5u_2^2u_1 + 2u_3^5u_2u_1^2 - u_3^5u_1^3 - u_3^3u_2^4u_1+ \\ & 4u_3^3u_2^3u_1^2 - 6u_3^3u_2^2u_1^3 + 4u_3^3u_2u_1^4 - u_3^3u_1^5)x_9x_7+ \\ & (-u_3^4u_2^4u_1 + 3u_3^4u_2^3u_1^2 - 3u_3^4u_2^2u_1^3 + u_3^4u_2u_1^4- \\ & u_3^2u_2^6u_1 + 5u_3^2u_2^5u_1^2 - 10u_3^2u_2^4u_1^3 + 10u_3^2u_2^3u_1^4- \\ & 5u_3^2u_2^2u_1^5 + u_3^2u_2u_1^6)x_7\end{aligned}$$
$$\begin{aligned}p_{14} = {} & (-u_3^5u_2^2u_1 + 2u_3^5u_2u_1^2 - u_3^5u_1^3 - 2u_3^3u_2^4u_1+ \\ & 8u_3^3u_2^3u_1^2 - 12u_3^3u_2^2u_1^3 + 8u_3^3u_2u_1^4 - 2u_3^3u_1^5- \\ & u_3u_2^6u_1 + 6u_3u_2^5u_1^2 - 15u_3u_2^4u_1^3 + 20u_3u_2^3u_1^4- \\ & 15u_3u_2^2u_1^5 + 6u_3u_2u_1^6 - u_3u_1^7)x_9x_8+ \\ & (-u_3^4u_2^3u_1 + 3u_3^4u_2^2u_1^2 - 3u_3^4u_2u_1^3 + u_3^4u_1^4- \\ & u_3^2u_2^5u_1 + 5u_3^2u_2^4u_1^2 - 10u_3^2u_2^3u_1^3 + 10u_3^2u_2^2u_1^4- \\ & 5u_3^2u_2u_1^5 + u_3^2u_1^6)x_9x_7+ \\ & (u_3^5u_2^3u_1 - 2u_3^5u_2^2u_1^2 + u_3^5u_2u_1^3 + u_3^3u_2^5u_1- \\ & 4u_3^3u_2^4u_1^2 + 6u_3^3u_2^3u_1^3 - 4u_3^3u_2^2u_1^4 + u_3^3u_2u_1^5)x_7\end{aligned}$$
$$\begin{aligned}p_{15} = {} & (-u_3^8u_2^3u_1^3 + 2u_3^8u_2^2u_1^4 - u_3^8u_2u_1^5 - 2u_3^6u_2^5u_1^3+ \\ & 6u_3^6u_2^4u_1^4 - 7u_3^6u_2^3u_1^5 + 4u_3^6u_2^2u_1^6 - u_3^6u_2u_1^7- \\ & u_3^4u_2^7u_1^3 + 4u_3^4u_2^6u_1^4 - 6u_3^4u_2^5u_1^5 + 4u_3^4u_2^4u_1^6- \\ & u_3^4u_2^3u_1^7)x_7^2+ \\ & (-u_3^7u_2^5u_1^3 + 3u_3^7u_2^4u_1^4 - 3u_3^7u_2^3u_1^5 + u_3^7u_2^2u_1^6- \\ & 2u_3^5u_2^7u_1^3 + 8u_3^5u_2^6u_1^4 - 13u_3^5u_2^5u_1^5 + 11u_3^5u_2^4u_1^6- \\ & 5u_3^5u_2^3u_1^7 + u_3^5u_2^2u_1^8 - u_3^3u_2^9u_1^3 + 5u_3^3u_2^8u_1^4- \\ & 10u_3^3u_2^7u_1^5 + 10u_3^3u_2^6u_1^6 - 5u_3^3u_2^5u_1^7+ \\ & u_3^3u_2^4u_1^8)x_7\end{aligned}$$
$$\begin{aligned}p_{16} = {} & (u_3^5u_2^2u_1 + 2u_3^3u_2^4u_1 + u_3u_2^6u_1)x_8+ \\ & (u_3^6u_2u_1 + 2u_3^4u_2^3u_1 + u_3^2u_2^5u_1)x_7+\end{aligned}$$

92. Creating S-polynomial from the pair (p_6, p_{19}).

    Forming S-pol of p_6 and p_{19}:

$$\begin{aligned}p_{89} = {} & (-u_3^3u_2^3u_1 + 2u_3^3u_2^2u_1^2 - u_3^3u_2u_1^3 - u_3u_2^5u_1+ \\ & 4u_3u_2^4u_1^2 - 6u_3u_2^3u_1^3 + 4u_3u_2^2u_1^4 - u_3u_2u_1^5)x_2+ \\ & (-u_3^4u_2^2u_1 + u_3^4u_2u_1^2 - u_3^2u_2^4u_1 + 3u_3^2u_2^3u_1^2- \\ & 3u_3^2u_2^2u_1^3 + u_3^2u_2u_1^4)x_1\end{aligned}$$

    Reduced to zero.

93. Creating S-polynomial from the pair (p_6, p_{20}).

    Forming S-pol of p_6 and p_{20}:

$$\begin{aligned}p_{90} = {} & (u_3^4u_2^2u_1 - u_3^4u_2u_1^2 + u_3^2u_2^4u_1 - 3u_3^2u_2^3u_1^2+ \\ & 3u_3^2u_2^2u_1^3 - u_3^2u_2u_1^4)x_2+ \\ & (u_3^5u_2u_1 + u_3^3u_2^3u_1 - 2u_3^3u_2^2u_1^2 + u_3^3u_2u_1^3)x_1\end{aligned}$$

    Reduced to zero.

94. Creating S-polynomial from the pair (p_6, p_{21}).

    Skipping pair p_6 and p_{21} because gcd of their leading monoms is zero.

95. Creating S-polynomial from the pair (p_6, p_{22}).

    Forming S-pol of p_6 and p_{22}:

$$\begin{aligned}p_{91} = {} & (u_3^6u_2^3u_1^2 - 2u_3^6u_2^2u_1^3 + u_3^6u_2u_1^4 + 2u_3^4u_2^5u_1^2- \\ & 8u_3^4u_2^4u_1^3 + 12u_3^4u_2^3u_1^4 - 8u_3^4u_2^2u_1^5 + 2u_3^4u_2u_1^6+ \\ & u_3^2u_2^7u_1^2 - 6u_3^2u_2^6u_1^3 + 15u_3^2u_2^5u_1^4 - 20u_3^2u_2^4u_1^5+ \\ & 15u_3^2u_2^3u_1^6 - 6u_3^2u_2^2u_1^7 + u_3^2u_2u_1^8)x_2+ \\ & (u_3^7u_2^2u_1^2 - u_3^7u_2u_1^3 + 2u_3^5u_2^4u_1^2 - 6u_3^5u_2^3u_1^3+ \\ & 6u_3^5u_2^2u_1^4 - 2u_3^5u_2u_1^5 + u_3^3u_2^6u_1^2 - 5u_3^3u_2^5u_1^3+ \\ & 10u_3^3u_2^4u_1^4 - 10u_3^3u_2^3u_1^5 + 5u_3^3u_2^2u_1^6 - u_3^3u_2u_1^7)x_1\end{aligned}$$

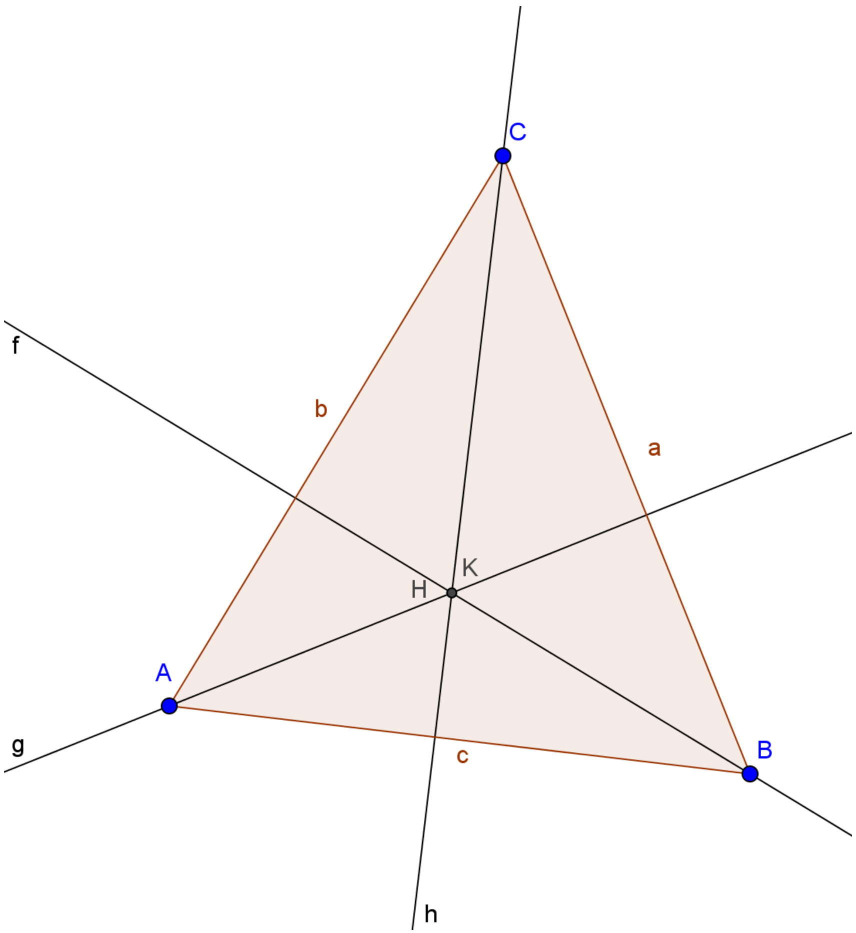
    Reduced to zero.

96. Creating S-polynomial from the pair (p_6, p_{23}).

    Forming S-pol of p_6 and p_{23}:

$$\begin{aligned}p_{92} = {} & (u_3^7u_2^4u_1^3 - 2u_3^7u_2^3u_1^4 + u_3^7u_2^2u_1^5 + 2u_3^5u_2^6u_1^3- \\ & 6u_3^5u_2^5u_1^4 + 7u_3^5u_2^4u_1^5 - 4u_3^5u_2^3u_1^6 + u_3^5u_2^2u_1^7+ \\ & u_3^3u_2^8u_1^3 - 4u_3^3u_2^7u_1^4 + 6u_3^3u_2^6u_1^5 - 4u_3^3u_2^5u_1^6+\end{aligned}$$

Avtomatsko dokazovanje - GeoGebra



H=Presečišče[g,h]

K=Presečišče [f,h]

j=Ekvivalenca[H,K]

Preveri[j]

true

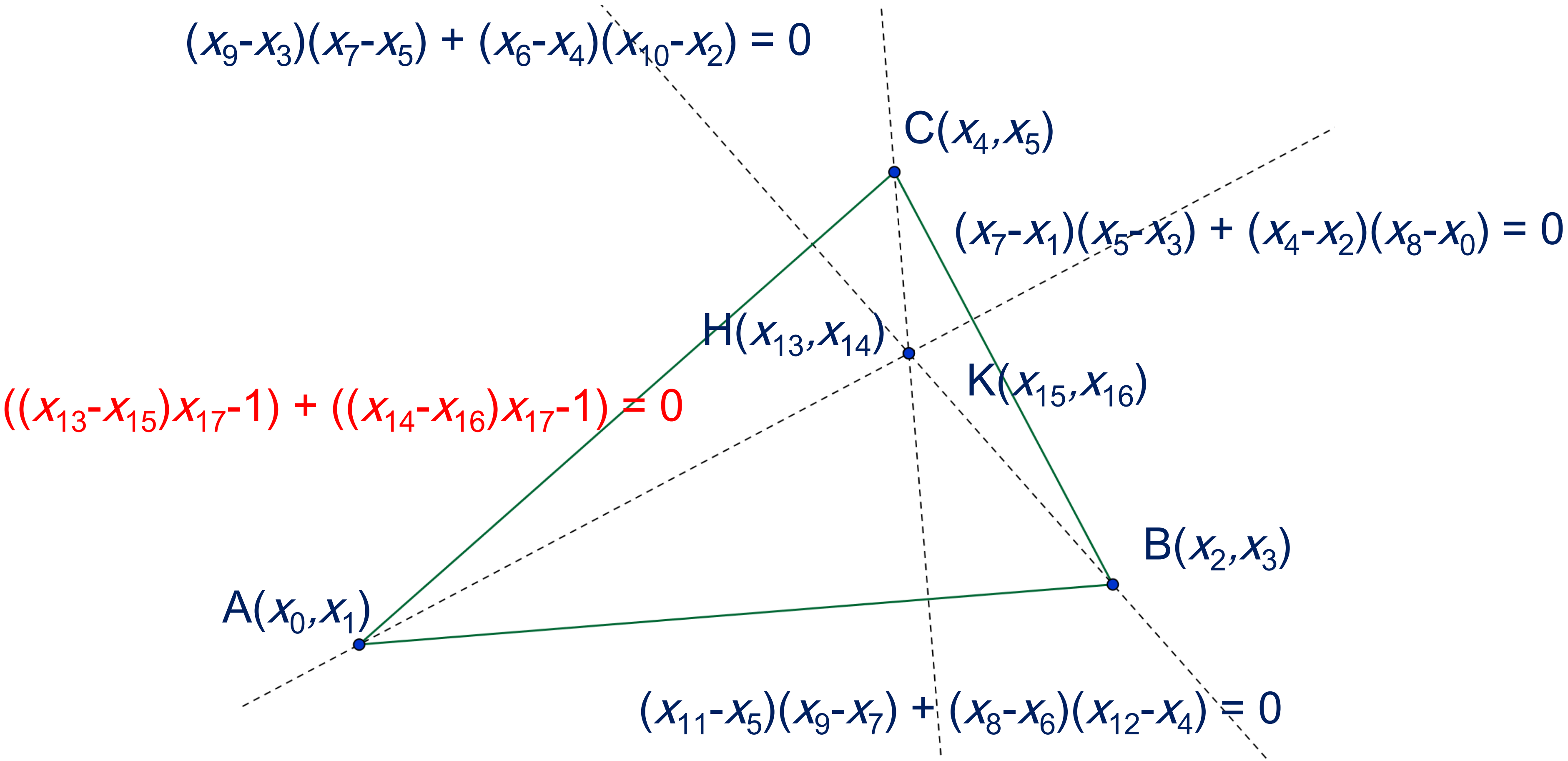
PodrobnostiDokaza[j]

{true,
{"SoKolinearne[A,B,C]"}}

Avtomatsko dokazovanje v geometriji

- Metoda ploščin (Chou, 1982)
- Algebrske metode (Wu 1977, Buchenberg, 1965)
- Metoda primerov (Hong, 1985)
- …

Algebrska metoda



Algebrska metoda (Wu 1977, Buchenberg, 1965)

pogoji

- $p_1(x_1,x_2,...x_k)=0$
- $p_2(x_1,x_2,...x_k)=0$
- ....
- $p_n(x_1,x_2,...x_k)=0$

posledica

$s(x_1,x_2,...x_k)=0$

$$\mathbf{s(x_1,x_2,...x_k)} = r_1(x_1,x_2,...x_k) \cdot \mathbf{p_1(x_1,x_2,...x_k)} + ... + r_n(x_1,x_2,...x_k) \cdot \mathbf{p_n(x_1,x_2,...x_k)}$$

Algebrajska metoda
(Wu 1977, Buchenberg, 1965)

pogoji

- $p_1(x_1, x_2, \ldots x_k) = 0$
- $p_2(x_1, x_2, \ldots x_k) = 0$
- ....
- $p_n(x_1, x_2, \ldots x_k) = 0$

posledica

$s(x_1, x_2, \ldots x_k) = 0$

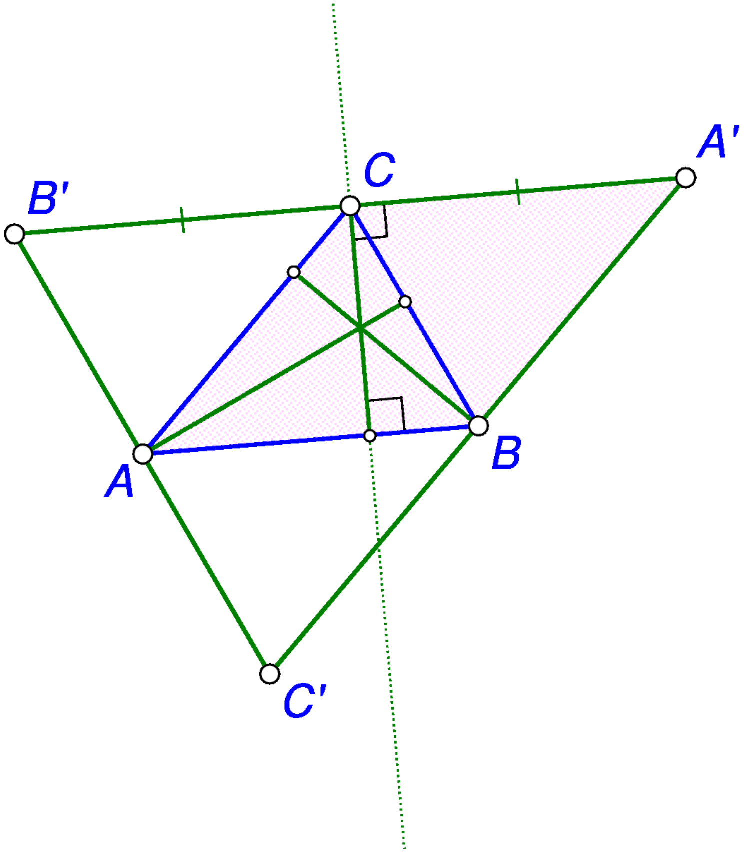
$$q(x_1, x_2, \ldots x_k) \cdot \mathbf{s(x_1, x_2, \ldots x_k)} =$$
$$r_1(x_1, x_2, \ldots x_k) \cdot \mathbf{p_1(x_1, x_2, \ldots x_k)} + \ldots + r_n(x_1, x_2, \ldots x_k) \cdot \mathbf{p_n(x_1, x_2, \ldots x_k)}$$

Avtomatsko dokazovanje in SŠ geometrija

- **Bo ATP postalo del šolske matematike?**
- Poglobljene obravnave preprostejših dokazov
- Tudi drugačne obravnave dokaza
- Zapis dokaza
- Potrebni in zadostni pogoji, posledice

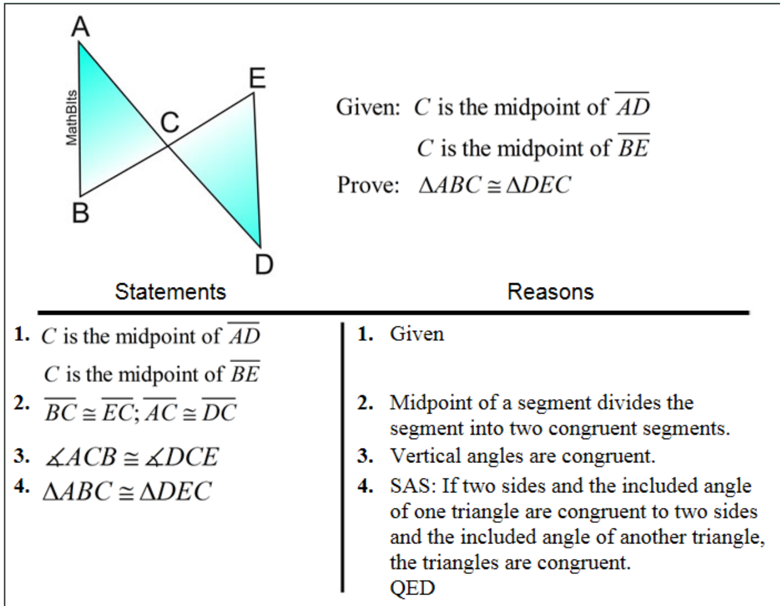
Avtomatsko dokazovanje in SŠ geometrija

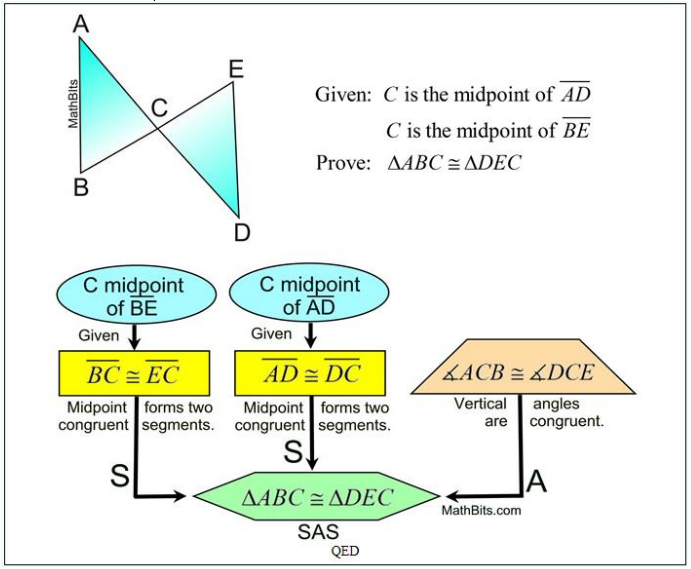
- **Bo ATP postalo del šolske matematike?**
- Poglobljene obravnave preprostejših dokazov
- Tudi drugačne obravnave dokaza
- Zapis dokaza
- Potrebni in zadostni pogoji, posledice



Avtomatsko dokazovanje in SŠ geometrija

- **Bo ATP postalo del šolske matematike?**
- Poglobljene obravnave preprostejših dokazov
- Tudi drugačne obravnave dokaza
- Zapis dokaza
- Potrebni in zadostni pogoji, posledice





Avtomatsko dokazovanje in OŠ geometrija

- Motivacija za dokaz (Jahnke, 2009; Prus… Herschowitz, Schwa… 2011;…)
- Neformalna utemeljevanja
- Preprosti dokazi
- Pojem dokaza

Hvala za pozornost