An example of good practice in self-discovery math teaching: nail boards and Pick's theorem

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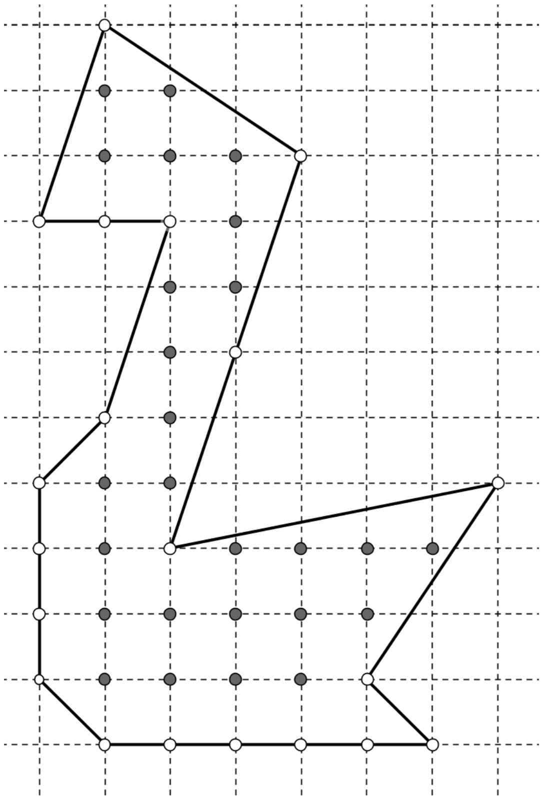
Abstract

Pick's theorem gives a simple, but usually unknown formula for the surface of polygons with all their vertices on a lattice. We describe in small different steps how to teach pupils to discover themselves this formula with the only required knowledge being the formulas for the surface of a rectangle and a triangle. We provide examples of differentiation how the teacher can work with the same material but on a different level of depth. We sketch the idea how this self-discovering approach can lead to insight at the pupils level of what a mathematical reasoning (or proof) is about.

Keywords: area of lattice polygons, self-discovery teaching, differentiation, mathematical thinking and proofs.

1. Introduction

A very simple and elegant formula for calculating the area of a lattice polygon is given by the formula of Pick[1]. All one has to do is count the lattice points inside the figure and on the border of the figure. With a simple calculation, this gives you enough information to calculate the precise area.



More precisely, <u>Pick's number</u> associated to a polygon is the sum of the number of lattice points completely inside the polygon and half of the number of the lattice points on the border of the polygon. <u>Pick's theorem</u> states that the area of such a polygon is one less than Pick's number associated to this polygon, times the area of the basic lattice square. This is true for all non-self-intersecting polygons with no holes and with all the vertices on the lattice.

Example. The figure shows a swan-like lattice polygon. There are 26 lattice points inside this figure (grey dots) and 20 lattice points on the border (white dots). Pick's number associated to this polygon is therefore 26 + 20/2 = 36. Pick's theorem states that the area is equal to 35 times the area of a basic lattice square. A basic square is the area of the (smallest) lattice square. So if you have a lattice point every cm, the area of the basic lattice square is 1 cm^2 and in that case the swan has an

[1] **George Alexander Pick** (Vienna 1859 – Theresienstadt concentration camp 1942) was an Austrian mathematician and is the author of the theorem, published in (Pick, 1899).

area of exactly 35 cm^2.

In this article, we describe an example how this formula can be used to give a good lesson in self-discovery math. Main objectives are the activity of pupils, and their (beginning of) understanding of mathematical processes. In that way, pupils will experience that mathematics is not about "the knowledge of formulas" but about "precise (but general) description of experiences". The relations described in formulas should become natural or evident, at least in simple situations.

Pupils or teachers who took this lesson, remembered even after several years this workshop. Especially the fact that they recalled the activity (working on a nail board or geoboard) and the fact they still related counting nails to calculate areas, convinces me of the effectiveness of this lesson.

Normally in this lesson, I let pupils work in groups of four and let them work independently under supervision. In § 5 you find an example of the written output each pupil received during this lesson.

If not mentioned otherwise, we will always assume the basic lattice square has area 1 cm^2.

2. Analysis of the self-discovery lesson with nail boards on Pick's theorem

In this section we describe some points of attention while performing this kind of lesson with the working sheets of § 5.

2.1 Working in groups

As a teacher you can divide your class into groups. I advise in this lesson to make homogeneous groups of pupils. That makes it easier to differentiate on the content: although everybody is working with the same material on the same subject, there are different levels of depth possible (e.g. step 3 or 7 of § 5, or § 3 to get further ideas).

Pupils in one group have to cooperate: they have to make different figures, put all conclusions on a sheet,… so they will (or have to) help each other.

As a teacher you just assist. This means in this case: look that they are really working on the assignment, help them when a group does not understand one or more steps in the process and look for counting errors especially during steps 1-4 of § 5. Since as a teacher, you already know Pick's theorem, you can easily see when they made an error in counting nails or in calculating areas.

2.2 Self-discovery

The purpose of this lesson is to get the pupils aware of the fact that mathematical formulas come in a natural way. Therefore, the emphasis lies not on the elegant formula but on the activity. Pupils should discover the formula more or less themselves as an 'obvious relation' they can read of their table of conclusions. The intellectual guidance as a teacher, mainly consists of the different steps you prepared for them in the text.

After doing this kind of lesson, one should have a discussion with the pupils talking about their experiences. I mostly recall another example of generalization of obvious thoughts: calculation rules for adding, subtracting…. since generalization is a basic idea in doing mathematics. Of course, while doing this kind of activity, you also train their ability of fulfilling different steps with great accuracy.

Thus it is important that pupils put their thoughts into concrete and exact words while working. So you ask not only for the intermediate results but they have to describe in words what and why they are working the way they do (steps 2, 5, 6, 7 of § 5).

2.3 Words are more important than the formula

Even the main result, Pick's theorem, is stated in words (step 5 of § 5), although we try to be precise. I do not think that pupils gain a better insight of this formula by introducing formulas or symbols. Therefore I do not see extra value in introducing N_i (or N_b) as the number of nails inside (or on the border of) the polygon when working with pupils. Nevertheless it is a good and difficult exercise to formulate with accuracy what the right conclusions are. This gives pupils the insight that its takes time to come to a statement that is 'mature'. Make sure you make them aware of this mathematical process, since most of the time a formula (in Belgium) is given as a fact, without experiencing the difficulties in order to get to such a statement.

2.4 Working sheets in Slovene (see § 5)

Some comments on the working sheets. In order to use for younger children, you have to use a little less text and more figures and symbols. It makes it also easier when every step starts at a new page.

Make sure that you give enough little steps to come to the conclusion but keeps every step non-trivial. This is a difficult balance when performing this kind of lesson. Don't be afraid of adapt the working sheets to the level of your pupils. It possibly takes some time as a teacher to have good balanced working sheets for the pupils you are working with.

As pointed out in § 2.2, make sure you leave some (more) space (than in the example) to let pupils reflect on their actions and put these into words. When pupils are not used to put their way of thinking into words, this may take some help. Draw their attention to the accuracy of their writings.

Working with the geoboard creates the possibility of a personal touch to this mathematical lesson. In this basic lesson, you see this in step 6d. You should really insist that they make a personal figure, not a boring triangle. The nail board gives them the opportunity to make something beautiful: a flower, a hockey stick… something in their daily interest. They will remember for a long time that they did this.

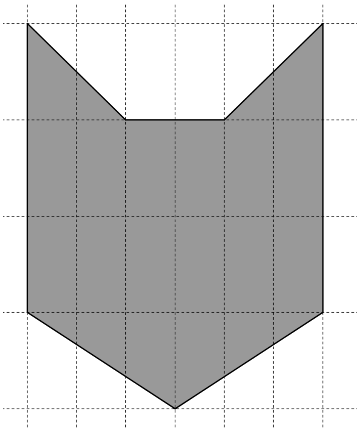
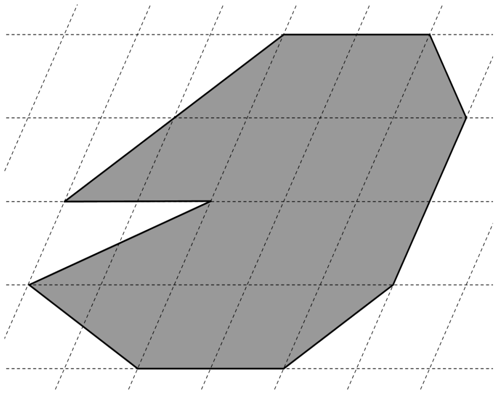
3. Beyond the standard formula

If you use this lesson at a latter age, it is possible to get your pupils acquainted with mathematical thinking and even with the ingredients of a proof.

3.1 Non-square lattices

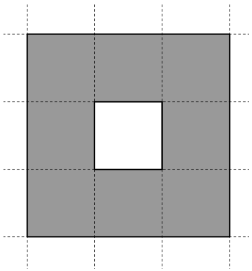
You normally work with a standard lattice: this means perpendicular axis and 1 cm between nails, so the basic lattice quadrilateral is a square of 1 cm^2. In step 7a-b of § 5, we suggest some questions concerning ratios. What happens if the distance between nails is multiplied by two? Then the area is multiplied by four. So in fact the Pick's theorem shows that the area of the lattice polygon is the area of the basic lattice square, multiplied by Pick's number minus one.

You can also change your lattice by making the basic lattice quadrilateral a rectangle (left figure) or even a parallelogram (right figure). The conclusion remains the same.

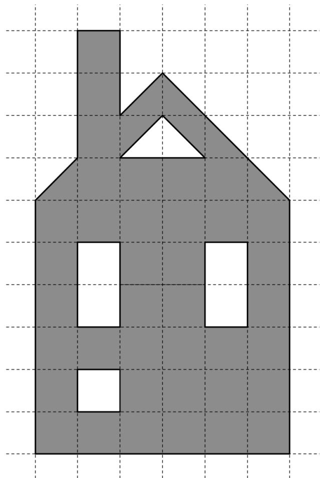
3.2 Figures with holes

In the same self-discovery way of working, you can give pupils to examine the following figure with a hole in it (step 7e of § 5).



Basically Pick's formula concerns a topological invariant. So let your students fill the table of conclusions (see § 5) for this figure, and notice that the original statement of Pick's theorem no longer holds. For the younger pupils, you can describe it as a figure with an inside and an outside border, so you need two non-intersecting elastic bands to create it.

They should detect by looking at the last two columns in the sheet "Table of conclusions" that they can easily adapt the formula for the area of figures with one hole. The conclusion should be that "minus 1" in the original formula seems to disappear. In other words, Pick's number associated to the polygon is equal to its area. So the number of nails inside the figure plus half of the number of nails on the border is now exactly the area of such figures.



A step further for the fast students will contain the situation of multiple holes in a lattice polygon. It is always a surprise for pupils that the number of holes, or in their language, the number of elastic bands you need to build your figure, determines the formula you should use for calculating the area.

In words, one can conclude that "*the area of a polygon with vertices on the nails of a nail board is equal to the sum of all nails completely inside the figure and half the number of nails on its border, minus 2 plus the number of elastic bands you need to make the figure on the board*." With the terminology of holes, this sounds like *"the area of a polygon with vertices on the nails of a nail board is equal to the sum of all nails completely inside the figure and half the number of nails on its border minus 1 plus the number of holes inside your figure."*

The latter formulation is closer to the original statement of Pick and therefore pupils prefer that one.

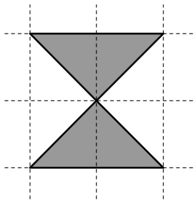
The formula is simple and beautiful. Nevertheless one should emphasize the mathematical way of thinking, which comes in a natural way. It consists of three steps:
1° one has a nice formula that seems to work properly;

2° one encounters a figure, where the formula does not work anymore;
3° one looks for adaptations of the original formula in order to state a more general/precise conclusion.

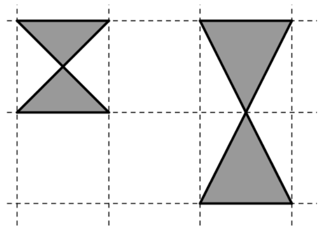
Next to that, pupils can get an idea of a real proof in this setting. Considering the figure with holes as a difference between lattice polygons, one gets the immediate result. Indeed, the polygon with outside border minus all polygons which form the holes, are all lattice polygons and therefore the original Pick formula holds for all these lattice polygons. This provides a waterproof argument (or proof) of the conclusion you already discovered by looking at the relationship between area and counting nails.

3.3 Self-intersecting figures

Other figures where the original formula does not hold any more, and which are makeable on the nail board with just one elastic band, are self-intersecting figures.

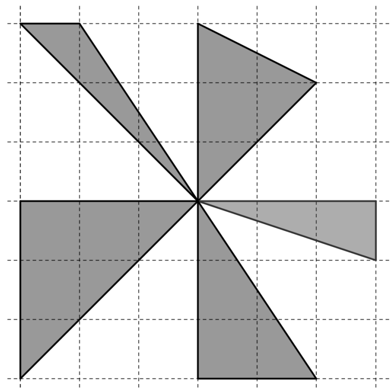


This intersection may happen exactly on a lattice point/nail (left figure), or not (two right figures).



For the left figure, one gets that the area equals Pick's number minus 3/2. For the right figures, there is no hope for a simple formula: both figures have the same Pick number since they both have 4 nails on the border. Nevertheless they have a different area.

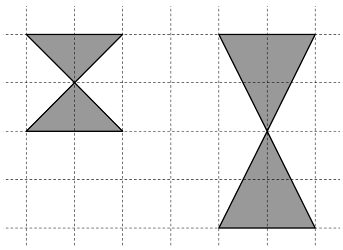
So pupils have to be very careful here. The easiest way to formulate the conclusion is the following one: the original formula of Pick's theorem has to be adjusted by subtracting half of the multiplicity of the intersection nail. The multiplicity equals the number of times that the nail is crossed when following the whole border when you start (and also end) in the intersection nail. Start and finish are not counted, so a non-intersection mail gets multiplicity zero! For the above figures, the multiplicity of the intersection nail equals one. For the figure on the right this multiplicity is equal to four.



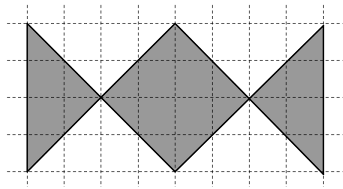
The upper left figure gives you 7 points on the border. This gives you a Pick number of 7/2. There is one intersection point of multiplicity one, so you should subtract 1 (from original Pick's formula) and a half (from the intersection point with multiplicity 1) from it. This gives you the correct area of 2 cm^2.

Notice that when intersection points are on the nails, you can see the figure as a sum of different lattice polygons where the original Pick's theorem holds. In this way, one understands that the intersection nails are counted more than once. For the example figure on the right side, the intersection point is counted five times when you see this as a sum of five lattice triangles, but also the "minus 1" occurs five times.

What do you have to do when intersection points are not on the lattice? At first sight no easy formula can be possible (see remark above). But when you think of § 3.1 the answer is relatively easy: just choose another lattice that is suited for your figure. ‘Suited’ here means that every intersection point lies on a lattice point. For the upper right figures, one can choose a lattice where the distance between nails is half the distance of the original one. Using the formula for self-intersecting polygons and using that the new basic lattice square is ¼ of the old one, you can again calculate the area by counting nails. Remark that it is always possible to choose a suitable lattice.



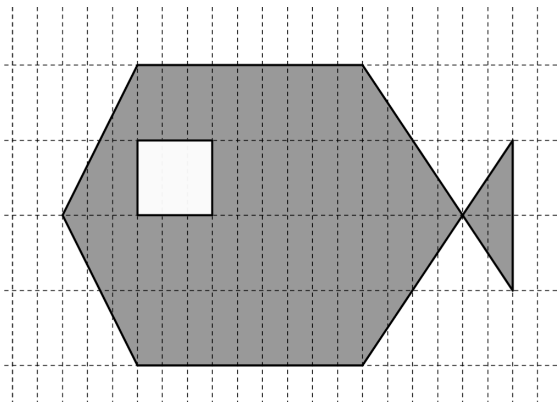
I usually leave the part of intersection points on the lattice for the gifted children. The part of intersection points outside the lattice is only used as a challenge for the very gifted.

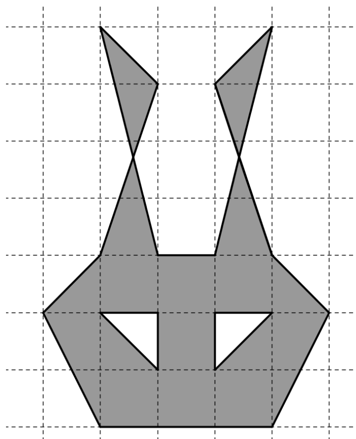


Again one can get to the more general case where there is more than one intersection point. Pick’s theorem can be adapted by subtracting half of the multiplicity of every intersection nail.

3.*4 Combining the previous*

Once you treated more than one situation of § 3.1 - 3.3 seperatly, you can combine them. This gives pupils to calculate areas of following figures just by counting nails and adjust Pick’s theorem in the right way.





Conclusion: the area of a polygon made by elastic bands on a nail board with only intersection points on nails, equals the area of the basic lattice quadrilateral times

Pick’s number (which is the sum of nails completely inside the figure and half of the nails on the border)

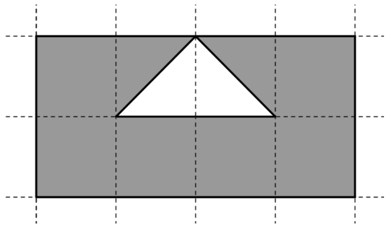
PLUS the number of holes

MINUS half the number of intersection nails (times their multiplicity)

MINUS one.

Almost equivalent and a little bit easier is: when a polygon has intersection nails, divide the figure into different subfigures each without intersection nails. The area of each subfigure can be calculated by Pick’s number PLUS the number of holes MINUS one.

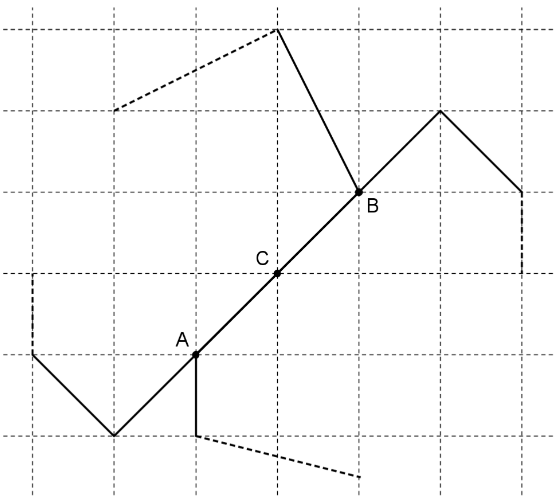
Remark that the second formulation cannot deal with the figure aside. The intersection nail cannot be removed by splitting the figure into two subfigures.



3.5 What is a proof all about?

The original theorem of Pick has been taken for granted, based on evidence. Nevertheless, once pupils have seen some of the simple counterexamples in the previous sections, you can ask them for a proof of the theorem. The right question to ask is: how should you build arguments in order to be absolutely sure that Pick's original theorem is true for non-intersecting polygons with no holes.

If you look in the literature, one finds a lot of different and elegant proofs (Dubeau and Labb, 2007), (Grünbaum and Shephard, 1993), (Pullman, 1979), (Tant, 2006), (Varberg, 1985). Nevertheless none of them is evident; most of the time it involves additivity of a certain function.
If you look at the first steps of the lesson (steps 1-3 of § 5) they may have an idea. First take the easiest polygons: rectangles with sides parallel to the lattice and rectangular triangles where both sides that form the right angle are again parallel to the lattice.
The formula of Pick can be checked for the rectangles, since it is both easy to calculate the area as well as to count the number of nails. Remark the triangles are just half of such rectangle and therefore Pick's theorem follows easily.



The main step is proving additivity. More precise: if two lattice polynomials share a (part of a) side, and Pick's theorem holds for both polygons, then it holds also for the polygon which is the union of both.
In the example both polygons share segment AB. You can see that every inside nail remains inside; nails on the border, remain on the border except for the nails on the segment AB (point C in the figure). These nails become inside nails in the great polygon. But this type of nails are no problem for Pick's formula since they are counted twice as half a nail when looking at the two smaller polygons separately and once as an inside nail when looking at the big polygon.

Points A and B play the key role. They are counted twice as a border nail when looking at the two small polygons separately, while they only should be counted once as a border nail of the greater polygon. Nonetheless this makes up for the fact that when adding Pick's theorem for the two polygons separately, you get twice the term "minus 1". Therefore one concludes that Pick's theorem still holds for the union of two polygons that share a segment.

Once you know the sum, you can also conclude that Pick's theorem still stands for the difference of two polygons sharing (part of) a side, where the smaller one completely lies in the bigger one.

The last step consists in the fact that every lattice polygon can be written as sum and difference of different basic rectangles and rectangular triangles as described in the first step (see step 3 of § 5 to get an idea). This goes in fact about tiling a lattice polygon, and remains at an intuitive level.

This kind of proof exercise is only useful for pupils with good math skills. Nevertheless, it contains a lot of exemplary information of how a mathematical proof works. They at least experience that, although the evidence of the formula is overwhelming, it still takes several non-trivial steps to actually be completely sure that it always holds. It this we can recognize three major steps:
1° check the statement in easy conditions;
2° move from there to more difficult situations;
3° try to proof that you can consider every situation when step 1 and 2 are known.

4. Conclusion

Pick's theorem can be used in class in a self-discovery way. Some teachers already use this theorem in class, most of the time emphasizing on calculating the areas of strange figures and creativity in building figures.

But the easiness and unfamiliarity of the formula creates the opportunity for discovering mathematical thinking. Pupils are often surprised they can discover formulas themselves and gain mathematical confidence out of it. Going beyond the standard formula, you can give pupils a glimpse of what is mathematical activity is all about: searching for more general statements when you encounter counterexamples by investigating several examples and trying to find an argument why it always works.

The nail board gives pupils the chance to do this investigation themselves and leaves some space for creativity (e.g. steps 6d or 7e in § 5). There are applets on the internet, but they might do all counting for you and are not capable of dealing with figures with holes or self-intersections (see [6]). Nevertheless other people developed similar material, e.g. [7] where you can find a geoboard applet that allows figures with holes.

Since there are a variety of figures where one has to adapt Pick's theorem, it is perfectly doable to differentiate in a lesson both in depth and in steps, while working with the same material. Motivate your pupils to make personal figures whenever they get the chance. This will enlarge their appreciation for math and give them more confidence.

I would just encourage teachers to incorporate lessons that focus on math thinking. Therefore I include an example of working sheets one can use in such a lesson. I hope you'll enjoy it, and just feel free to contact me for advice or to share your feedback.

5. Appendix: an example of working sheets to use in class (in Slovene)

See added Word-files, or look at p. 649-655 in the conference proceedings
http://www.zrss.si/pdf/zbornikprispevkovkupm2012.pdf

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